A canonical basis approach to unitarity calculations in Yang-Mills theories.

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A Canonical Basis Approach to Unitarity Calculations in Yang-Mills Theories

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Abstract

A Canonical Basis method is presented for computing the cut-constructible parts of loop amplitudes in massless Yang-Mills theory by generalised Unitarity. The method relies upon constructing a canonical basis of general cut solutions and constructing the amplitude by identifying the integral coefficients with elements of the canonical basis, thereby avoiding repeated integration of similar integrand structures. The method provides closed, rational, fully analytic expressions for the integral coefficients. As an application the method is applied to compute the previously unknown NMHV partial amplitudes of the 7-gluon $\mathcal{N} = 1$ chiral multiplet loop, and is then extended to rederive the cut-constructible terms of the NMHV 6-gluon partial amplitudes with a complex scalar loop.

DECLARATION:

This work has not previously been accepted in substance for any degree and is not being concurrently submitted in candidature for any degree.

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This work is the result of my own investigations except where otherwise stated. Other sources are acknowledged by explicit references. A bibliography is appended.

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The work presented in Chapters 3 and 4 was performed in collaboration with Professor David Dunbar and Doctor Warren Perkins, and was published in:

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The work presented in Chapter 5 is entirely the result of my own investigations.

Contents

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Chapter 1

Introduction and Motivation

1.1 Introductory Remarks

Ever since its theoretical development from Quantum Mechanics over 50 years ago, the theoretical framework of Quantum Field Theory has proven enormously successful in unlocking the mysteries of subatomic physics. In this theoretical framework, the techniques of perturbation theory in particular have proven spectacularly effective in yielding useful predictions of physically observable quantities which can then be verified experimentally using particle colliders. Despite the limitations one might expect due to the specific limited regime of weak coupling in which perturbation theory is applicable, it has nonetheless allowed predictions to be made allowing experimental verification of almost all of the component particles of the Standard Model, and although it is important to remember that there are significant theoretical obstacles outstanding to which perturbation theory cannot be applied, notably to strongly-coupled theories, still perturbative techniques are at the forefront of modern physics, providing predictions for the likely mass range of the final Standard Model ingredient, the Higgs boson and for predicted elements of many of the plausible theories of possible BSM behaviour, such as the Minimally Supersymmetric Standard Model (MSSM).

Although with the possibility of new physics discoveries opened by the long-awaited start of experimental physics at the LHC, and with the ongoing attempts to unify gravity with gauge theory using String Theory, it is possible that Quantum Field Theory may within a few years give way to some new theoretical framework, it is at present still a highly current and prolific theoretical perspective with regards to developing physics which can be tested with the collider technology likely to be available in the near future. However, the outstanding challenges to searching for new physics at the LHC are formidable; at time of writing, it appears that the formidable experimental obstacles have finally been overcome, and as such it is possible that the main obstacles in the near term to progress will come from the theoretical side. In particular, there is a unique need for accurate perturbative calculations of NLO and NNLO processes with many external particles which could largely be avoided in earlier collider generations. This is largely due to the problems associated with background arising due to the nature of proton-proton collisions and the large amount of QCD background processes they entail. For example, a significant hurdle to the search for the Higgs boson arises due to the fact that the normal expectation that NLO processes will be dominated by the LO contribution is often not the case due to the coupling of the Higgs to loops of top quarks and massive vector bosons. In addition, the purely QCD background to many processes is large and difficult to accurately compute, and as such most of the processes being considered as likely Higgs detection channels are those arising from the weak sector (see table 1.1).

These calculations of loop-dependent processes present a major phenomenological obstacle; indeed, much of the progress in QFT for the last few decades has revolved around the problem of computing scattering amplitudes for processes containing particle loops, i.e. containing a closed loop of indices in the transition matrix element. Since such a closed loop results in a momentum circulating which is not constrained by the external momenta via momentum conservation, we thus must integrate over all possible contributions from this momentum, giving rise to loop integrals.

QFT contains a framework (renormalization) for obtaining meaningful expressions from such integrals, but although this means a calculation of a UV-divergent loop process is in principle *possible*, actually computing a non-trivial loop amplitude in QCD is a formidable computational problem both due to the non-trivial nature of evaluating the loop integrals and to the typically large numbers of Feynman diagrams loop processes tend to give rise to. As such, until recently calculations of loop amplitudes with many external particles tended to be performed individually as a phenomenological need arose, and almost invariably employing some degree of manipulation to reduce either the number of Feynman diagrams or simplify the integrals themselves; perhaps the best known example of such an approach is the Passarino-Veltman reduction technique [1], which together with its successors is still relevant today and is discussed in Chapter 2.

This state of affairs persists today, i.e. the phenomenological demands of the LHC have called for a great deal of effort on the theoretical side to develop methods to compute the required loop amplitudes in a reasonable time frame. Indeed, this process is ongoing; the 2005 Les Houches Phenomenology Working Group report [2] identified a wishlist of 8 NLO processes as priorities for accurate predictions of LHC background processes. This wishlist was updated by the 2008 [3] and 2010 [4] Les Houches reports with an additional 4 NLO and 4 NNLO processes. The 2010 wishlist is given in table 1.1.

It is worth noting that all of the wishlist processes contain 5, 6 and in some cases even 7-point loops. It is clear therefore that the development of new techniques and refinement of existing methods for computing such complicated loop amplitudes is of considerable importance for progress in fundamental physics. In addition, the

NLO processes calculated since Les Houches 2005
1: $pp \to VVjet$ [5, 6, 17]
2: $pp \to H + 2jets$ [7, 8]
3: $pp \to VVV$ [9, 10, 31]
4: $pp \to t\bar{t}b\bar{b}$ [11, 32]
5: $pp \to V + 3jets$ [20, 12]
NLO processes still outstanding
6: $pp \to t\bar{t} + 2jet$
7: $pp \to VVb\bar{b}$
8: $pp \to VV + 2jet$
9: $pp \to b\bar{b}b\bar{b}$
10: $pp \to V + 4jet$
11: $pp \to Wb\bar{b}j$
12: $pp \to t\bar{t}t\bar{t}$
NNLO processes
13: $pp \to W^{*}W^{*}$
14: $pp \to t\bar{t}$
15: $NNLO$ to VBF and $Z/\gamma + jet$
16: $NNLO$ QCD + NLO EW for W/Z

Figure 1.1: The 2010 Les Houches wishlist of phenomenologically interesting loop processes

difficulty of accurately estimating the QCD background means that processes which would be strongly dominated by the background are unfortunately difficult to use as meaningful experimental tests. This suggests a long-term benefit to developing methods to accurately compute QCD background amplitudes in addition to the short-term phenomenological goals identified above.

1.2 Progress in Feynman techniques

As an illustration of the poor scaling of Feynman techniques in Yang-Mills theory, one can compute the number of Feynman diagrams contributing at tree level to the process $gg \to ng$, a set of results collated by Mangano and Parke [13].

As can be seen, the progression is extremely rapid in n, leading to the severe proliferation in contributing terms for large n. This behaviour at tree level is only compounded at loop level, where one also has rapid proliferation of form factors appearing during reduction of the tensor integrals. This provides a severe practical

n	2	3	4	5	6	7	8
# of diagrams	4	25	220	2485	34300	559405	10525900

Figure 1.2: Illustration of proliferation of number of Feynman diagrams with number of external legs for tree-level gluon scattering [13]

limitation on the evaluation of loop amplitudes containing many external particles analytically using Feynman techniques. As a consequence of this contemporary applications of Feynman diagram methods to loop amplitudes of 5 or more external particles are at least partially numerical and automated. Recent research in the area largely expands upon the selection of automated packages which exist for performing calculations of NLO amplitudes with up to four external particles [14, 15]. Using such techniques various collaborations have been able to provide calculations of processes from the Les Houches wishlist [5, 8, 9, 10].

A notable approach in the direction of fully automating computation of multileg loop amplitudes via Feynman diagrams is the Golem95 package developed by Binoth, Guillet, Heinrich, Pilani and Reiter [16], which fully automates the process of evaluating higher point loop amplitudes numerically using Feynman diagram techniques. The approach used is to evaluate the tensor integrals appearing in the calculation numerically in order to avoid the algebraic proliferation which would otherwise occur. The approach has been applied to calculate in particular the process $pp \to ZZ + jet$ [17].

1.3 Overview of Existing Unitarity Implementations

Due to the inherent problems associated with computing loop amplitudes using Feynman techniques, there has been a great deal of progress in developing techniques to compute such amplitudes by alternative methods. By far the most promising alternative is the Unitarity approach; several implementations of the method have been developed, some of which have been automated. Perhaps the most clear indication of the method's promise is the fact that it is the only approach besides Feynman techniques to have succesfully been applied to partially or completely calculate processes from the Les Houches wishlist (see table 1.1), as discussed in the respective sections for the relevant implementations.

Although the complexity of a Unitarity calculation obviously does still rise with the number of external legs, this scaling is nowhere near as dramatic as with Feynman methods. In 4-dimensional generalised Unitarity, the leading order scaling in the number of cuts to compute for an n-point amplitude is polynomial in n. Clearly this gives much superior scaling in number of Unitarity cuts as the number of external particles increases compared to the scaling of the number of Feynman diagrams to compute the equivalent amplitude. To make this more clear, we can explicitly calculate the number of possible independent momentum channels which can contribute cut constructible terms to a colour-ordered Yang-Mills n-gluon partial amplitude, i.e. the maximum possible number of individual cuts which can appear, the results of which are shown in table 1.3.

n	Bubbles	Triangles	Boxes
4	2	4	1
5	5	10	5
6	9	20	15
7	14	35	35
8	20	56	70
9	27	84	126
Progression	$\frac{n}{2}(n-3)$	$\frac{n}{6}(n-2)(n-1)$	$\frac{n}{24}(n-3)(n-2)(n-1)$

Figure 1.3: Scaling of number of Unitarity cuts for gluon 1-loop amplitudes

Of course, this is only the scaling for the cut constructible parts. In addition while this analysis gives the total number of cuts, it does not account for the increase in difficulty of the cuts themselves once the component trees are NMHV or higher in structure, since such trees typically increase rapidly in complexity as the number of external legs increases. On the other hand it also does not account for the typically large number of cuts which vanish due to having no permitted helicity configurations for a given partial amplitude. It is also worth noting that the worst scaling behaviour appears in the purely algebraic quadruple cuts, with the double cuts (and thus the hardest-to-compute cut integrals) scaling only as n^2 to leading order.

The advantages in scaling behaviour compared to Feynman methods is balanced to an extent by the necessity to compute the non-trivial integrals appearing in the double and triple cuts. Developing methods for computing such integrals efficiently has been perhaps the most important obstacle in applying most Unitarity implementations to calculations useful amplitudes. There has been a great deal of theoretical progress on this problem in recent years, and a number of methods have been developed for calculating cut integrals systematically. The remainder of this chapter will therefore discuss and compare the most prominent methods, and the significant results they have yielded.

1.3.1 Direct Parametrization

Most of the modern implementations of Unitarity rely upon some method of manipulating the cut integrals into a form in which they can be more easily evaluated. The direct parametrization technique developed by Forde [19] is just such a method. It relies upon solving the difficult cut integral over the Lorentz-invariant phase space measure by substitution, by reparametrizing it in terms of a more conventional integration parameter. For example, for a triple cut in 4 dimensions, one can rewrite the loop momentum in terms of a single complex variable t, since the triple cut integral must necessarily be a one-dimensional integral. With an appropriate choice of $l(t)$, then, the remaining integral can be solved as a contour integral over t, and one thus obtains a solution consisting of the sum of the residues of all poles contained within the contour, plus those parts of the integrand not containing poles,

$$\int d^4l \prod_{i=0}^{2} \delta(l_i^2) A_1 A_2 A_3 = \int d^4l \prod_{i=0}^{2} \delta(l_i^2) \left([\mathrm{Inf}_t A_1 A_2 A_3](t) + \sum_{\{j\}} \frac{\mathrm{Res}_{t=t_j} A_1 A_2 A_3}{t - t_j} \right) . \quad (1.3.1)$$

Since taking the residue of a pole in t has the effect of eliminating it from the integral, this thus results in a term which is frozen out of the integral, and thus the residue terms can thus be identified as the box coefficient content of the triple cut. The triangle coefficient thus arises purely from the Inf_t term. This term is given by some polynomial in t [18]. The method relies upon choosing a parametrisation of l_μ in terms of t such that all integrals over positive powers of t vanish, resulting in the expression for the triple cut,

$$\int d^4l \prod_{i=0}^{2} \delta(l_i^2) A_1 A_2 A_3 = f_0 \int dt J_t + \sum (boxes) . \quad (1.3.2)$$

Since the integral over t is simply the scalar triangle integral, the term f_0, given by the first term in the expansion of the integrand in t at infinity, can be identified as the coefficient of the scalar triangle integral,

$$c_{ij} = -[\mathrm{Inf}_t A_1 A_2 A_3]|_{t=0}\,. \tag{1.3.3}$$

The application to the double cuts follows the same principle. Since the 4-dimensional double cut has two powers of loop momentum eliminated by the cutting process, the loop momentum must be parametrized in terms of two complex parameters instead of one. The contour integration thus yields

$$\begin{aligned}\int d^4 l \prod_{i=0}^{1} \delta(l_i^2) A_1 A_2 = \int d_t d_y J_{t,y} \Bigg(&[\mathrm{Inf}_t[\mathrm{Inf}_y A_1 A_2](y)](t) + [\mathrm{Inf}_t \sum_{\{j\}} \frac{\mathrm{Res}_{y=y_j} A_1 A_2}{y - y_j}](t) \\ &+ \sum_{\{l\}} \frac{\mathrm{Res}_{t=t_l}[\mathrm{Inf}_y A_1 A_2](y)}{t = t_l} + \sum_{\{j\},\{l\}} \frac{\mathrm{Res}_{t=t_l} \frac{\mathrm{Res}_{y=y_j} A_1 A_2}{y=y_j}}{t = t_l} \Bigg)\,. \end{aligned} \tag{1.3.4}$$

Once again the double residue term effectively freezes the remaining two integration orders, and this piece can be identified as consisting purely of boxes. One might expect the two single-residue terms to consist purely of triangle coefficients, however this is not the case as due to the different loop momentum parametrization for the double cut, the integrals over non-zero powers of y do not vanish. Instead one must explicitly construct all contributing triple cuts which can be obtained by inserting a propagator into the double cut. Specifically one first evaluates the double Inf term, then performs the triple cuts using the same momentum parametrization used for the double cut to obtain the bubble contribution from the single residue terms. The full bubble coefficient is thus,

$$b_j = -i[\mathrm{Inf}_t[\mathrm{Inf}_y A_1 A_2](y)](t)|_{t\to 0, y^m \to \frac{1}{m+1}} - \frac{1}{2}\sum_{C_{tri}} [\mathrm{Inf}_t A_1 A_2 A_3](t)|_{t^j \to T(j)}\,, \tag{1.3.5}$$

where $T(j)$ is the known integral over t of t^j.

The direct parametrization method has found particular application using the "Unitarity Bootstrap" approach of Bern, Dixon and Kosower [18], whereby one computes a 1-loop amplitude by using Generalized Unitarity to evaluate the cut-constructible parts, and on-shell recursion [70, 71] to compute the rational terms not yielded by 4-dimensional Unitarity. This approach has been automated in the form of the BlackHat C++ package developed by the Berger, Bern, Dixon, Forde, Ita, Kosower collaboration [50] in order to provide an automated package for evaluating interesting large-n scattering processes efficiently, in combination with BCFW on-shell recursion to evaluate the rational, non-cut-constructible terms. The BlackHat package has shown great promise with regard to providing accurate numerical predictions for the LHC. After initially being applied to rederive part of the 6-gluon scalar loop derived by Ellis, Giele and Zanderighi [82], the package has since been used in combination with the SHERPA event generator [26] to compute the processes contributing to the $pp \to W + 3jet$ [20] and the $pp \to Z, \gamma + 3jet$ [21] cross sections. In addition to the BlackHat results the Bootstrap approach has also been used to compute the case of a 1-loop n-gluon amplitude with a "split-helicity" configuration for all n [22], and more recently to compute the remaining unknown NLO contributions to the process $H \to 4parton$ [23, 24, 25].

1.3.2 OPP Integrand Reduction

The Integrand Reduction approach to Unitarity developed by Ossola, Papadopoulos and Pittau [27], is unusual in that while it relies upon identifying the cut-constructible coefficients of the basis integrals, it avoids entirely the problem of evaluating the actual cut integrals, instead proceeding from the most general expression of the integrand and attempting to reduce it to an expression free from dependence upon the loop momentum. The method builds upon the earlier result of del Aguila and Pittau [28], who showed that the integrand of any m-point one-loop amplitude with loop momentum q could be written in the form,

$$A(\bar{q}) = \frac{N(q)}{\bar{D}_0 \bar{D}_1 \cdots \bar{D}_{m-1}}, \tag{1.3.6}$$

where the bar notation denotes a d-dimensional rather than 4-dimensional momentum, the D's represent the loop propagators, and the numerator $N(q)$ has the form

$$\begin{aligned}
N(q) = & \sum_{i_0<i_1<i_2<i_3}^{m-1} \left[d(i_0 i_1 i_2 i_3) + \tilde{d}(q; i_0 i_1 i_2 i_3)\right] \prod_{i \neq i_0, i_1, i_2, i_3}^{m-1} \bar{D}_i \\
& + \sum_{i_0<i_1<i_2}^{m-1} \left[c(i_0 i_1 i_2) + \tilde{c}(q; i_0 i_1 i_2)\right] \prod_{i \neq i_0, i_1, i_2}^{m-1} \bar{D}_i \\
& + \sum_{i_0<i_1}^{m-1} \left[b(i_0 i_1) + \tilde{b}(q; i_0 i_1)\right] \prod_{i \neq i_0, i_1}^{m-1} \bar{D}_i \\
& + \sum_{i_0}^{m-1} \left[a(i_0) + \tilde{a}(q; i_0)\right] \prod_{i \neq i_0}^{m-1} \bar{D}_i \\
& + \tilde{P}(q) \prod_{i}^{m-1} \bar{D}_i \,,
\end{aligned} \tag{1.3.7}$$

i.e. that the integrand numerator consists of a sum of loop momentum-independent objects d, c, b and a, with some so-called "spurious" terms $\tilde{d}$, $\tilde{c}$, $\tilde{b}$, $\tilde{a}$ and $\tilde{P}$, which retain dependence upon the loop momentum. These terms are present only due to the construction of the 4-dimensional numerator $N(q)$ in the d-dimensional integral; they obey the property that they vanish in total in integration over the full loop measure $\int d^d\bar{q}$. This is achieved by cancellation with the rational pieces which are considered to arise from the separation of the $\tilde{q}$ dependent piece of the numerator, $\tilde{q}$ being the $(d-4)$ dimensional part of the loop amplitude. Since the OPP method allows the spurious pieces to be constructed from the "top-down", this property can thus be used to obtain the rational parts of the amplitude. Meanwhile, since the "spurious" terms vanish in the overall integral, the loop-momentum-independent terms multiplying scalar integrals can be identified as the coefficients of the integral basis.

The method isolates the spurious terms by choosing a particular parametrization for the loop momentum, and constructing all possible q-dependent tensors permitted by renormalizability. Applied to the general four-point numerator, one obtains

$$N^{(3)}(q) = d(0123) + \tilde{d}(0123)Tr[(\not{q}+\not{p}_0)\not{l}_1\not{l}_2\not{k}_3\gamma_5] + \sum_{i=0}^{3}\mathcal{O}(\bar{D}_i) + \mathcal{O}(\tilde{q}^2)\,, \tag{1.3.8}$$

where l_1, l_2 and k_3 are terms arising from the momentum parameterization. Since the first term is q-independent, and the third and fourth terms are not a true 4-point function due to the appearance of propagators and a contribution to the rational term, the second term can be identified as the 4-point spurious term. A similar process can be used to extract the exact q-dependence of the three, two, and one point spurious terms. The 0-point like term vanishes in the renormalizable gauge.

The next step is to extract the integral coefficients. In principle this could be done by computing a general numerator and using equation (1.3.7) to solve the equation for the coefficients at a sufficiently large number of values of q, however this in general would be an impractically large system of equations and the actual OPP implementation relies upon selecting values of q at which one or more denominators vanish, simplifying the system of equations. The process is iterative; a momentum is first chosen such that $D_0 = D_1 = D_2 = D_3 = 0$, isolating the 4-point terms and allowing the coefficient to be solved with only two solutions for q. With d and $\tilde{d}$ known, one can then use the constraint $D_0 = D_1 = D_2 = 0$ to solve for the 3-point coefficients in terms of the spurious terms and the 4-point terms; the process is repeated to extract b and a.

The OPP reduction technique has been automated in the CutTools package [29]. This package has been applied to give numerical evaluations first of the massless $2\gamma \to 4\gamma$ QED loop [30] as a proof of concept, followed by several important elements of the Les Houches wishlist, namely the process $pp \to VVV$ [31], the process $pp \to t\bar{t}b\bar{b}$ [32] and a partial computation of the $pp \to t\bar{t} + 2jet$ [33]. More recently the method has been combined with d-dimensional techniques in the SAMURAI Fortran90 package [34].

1.3.3 Spinor Integration

The Spinor Integration method developed by Britto and Feng arose ultimately out of the observation of Witten in 2003 that massless Yang-Mills theory can be interpreted as a twistor string theory [76]. In particular in that paper Witten observed that in transforming to twistor space, Yang-Mills MHV amplitudes become localised on a straight line in twistor space. This work was expanded upon with the discovery of the holomorphic anomaly [35], namely that although the tree amplitudes, and the rational coefficients of scalar integrals in a loop amplitude, must be annihilated by the twistor space collinearity operator,

$$F_{ijk;\dot{a}} = \langle i\, j\rangle \frac{\partial}{\partial\tilde{\lambda}_k^{\dot{a}}} + \langle j\, k\rangle \frac{\partial}{\partial\tilde{\lambda}_i^{\dot{a}}} + \langle k\, i\rangle \frac{\partial}{\partial\tilde{\lambda}_j^{\dot{a}}} \,. \tag{1.3.9}$$

The full cut integral is not annihilated when this operator acts upon three gluons in an MHV amplitude in the cut integrand. This observation was exploited [36] to develop an efficient method for evaluating loop amplitudes in $\mathcal{N} = 4$ super Yang-Mills, by considering how both the scalar integrals and the rational coefficients decompose under the action of the collinearity operator on a given cut integral, and thus writing the cut integral in a form in which the coefficients can be identified.

Since one-loop amplitudes in $\mathcal{N} = 4$ SYM consist purely of box integrals, the method was superseded by the development of Generalized Unitarity, which allows box coefficients to be computed straightforwardly from quadruple cuts in 4 dimensions. However, the method was later extended to develop the more general technique of Fermionic Integration which could be applied to compute the cut-constructible parts of gluon amplitudes with $\mathcal{N} = 1$ chiral and complex scalar loops [37]. The method again relies upon the localisation of tree amplitudes on a line in twistor space; one obtains the bubble and triangle coefficients from double cuts by rewriting the measure of the cut integral in terms of spinors, λ and $\tilde{\lambda}$, defined by $l_{a\dot{a}} = t\lambda_a\tilde{\lambda}_{\dot{a}}$.

The cut is thus reparameterised in terms of two integration parameters in a similar fashion to the Forde direct parametrization method, the difference being that while the t integration is a conventional integral, the integration over the fermionic parameters λ and $\tilde{\lambda}$ is done over the diagonal defined by $\tilde{\lambda} = \bar{\lambda}$. The cut integral thus becomes

$$\int_0^{\infty} d^4 l \delta^{(+)}(l^2) f(l) \to \int dt\, t \int \langle \lambda\, d\lambda \rangle\, [\tilde{\lambda}\, d\tilde{\lambda}] f(\lambda, \tilde{\lambda})\,. \tag{1.3.10}$$

After performing the t-integral, the integrand has the form,

$$\frac{1}{\langle l|P_{cut}|l]^n} \frac{\prod [a_i\, l] \prod \langle b_i\, l \rangle \prod \langle l|Q_k|l]}{\prod [c_i\, l] \prod \langle d_i\, l \rangle \prod \langle l|O_k|l]}, \; O_k \neq P_{cut},\; O_k^2 \neq 0\,. \tag{1.3.11}$$

This can then be split into a sum of simpler terms using the same partial fraction splitting identity used in chapter 3 of this thesis, equation (3.2.6). The result is a series of different classes of fermionic integrals which can be evaluated individually by rewriting them as a total derivative, and reading off the residues of the poles.

The method was further applied to the problem of computing the cut constructible parts of gluonic amplitudes with a complex scalar circulating in the loop [38]. In the process it was applied to rederive the NMHV 6-gluon scalar loop previously solved by Ellis, Giele and Zanderighi [82]. The spinor integration method has been further combined with d-dimensional Unitarity techniques [39] to allow computation of rational parts arising from $\mathcal{O}(\epsilon)$ terms, and has been generalized to apply to massive particles [40].

1.3.4 D-Dimensional Unitarity

It is a well-known property of Quantum Field Theory that the loop integrals appearing in amplitudes are UV divergent in 4-dimensions. In order to allow the integrals to be computed and the amplitudes evaluated, one must introduce some regularization scheme, generally by making some unphysical continuation of the theory into a region in which the integrals converge, and then regaining the physical amplitude by taking the limit back to the case where the unphysical abstraction is not present, after one has eliminated the UV-divergences appearing in the theory by renormalization, i.e., accounting for the difference between the "bare" physical parameters of the theory and the true, physical parameters, a process known as renormalization.

Although various regularization schemes exist, the most commonly used scheme in QCD is the dimensional regularization scheme [41]. In this scheme, one considers the loop integral not to be an integration over a 4-dimensional loop momentum but in an abstract number of dimensions D, where $D = 4 - 2\epsilon$, where ϵ is considered to be some small parameter. When the loop integrals are performed, the divergence typically manifests in the form of a $\frac{1}{\epsilon}$ or $\frac{1}{\epsilon^2}$ pole.

When the Unitarity method is used to find the coefficients of an integral basis such as that described in chapter 2, it is important to note that though the basis integral must be dimensionally regularized and thus contain poles in ϵ, the coefficients can be well defined in 4-dimensions. In the process of performing Passarino-Veltman reduction of n-point tensor integrals, one obtains coefficients which consist of some rational expression defined in terms of the four-dimensional external momenta, plus some $\mathcal{O}(\epsilon)$ term encapsulating the d-dimensionality [73]. Thus, the Unitarity cut integrals themselves can be performed in 4-dimensions to obtain the coefficients. In particular, since the particle momenta, spinors and polarization vectors are well defined in 4 dimensions, one can take full advantage of computational tools such as the spinor-helicity formalism (discussed in detail in chapter 2).

This is however not the only way the Unitarity cuts can be performed. It is possible to instead consider D-dimensional loop momentum cuts, conventionally done by treating the massless D-dimensional loop particles as massive 4-dimensional particles [42]. In the 2008 paper by Giele, Kunszt and Melnikov [44], this D-dimensional approach was applied as a numerical implementation of the Unitarity method. The method relies upon identifying that the a dependence upon the dimensionality can enter the integral from two sources: The dimensionality of the integration itself, denoted D, and the dimensional dependence arising from closed loops over metric tensors and Dirac matrices, denoted D_S, which can be at most linear for a 1-loop integral. The implementation differs significantly from 4-dimensional Unitarity; in particular, since the cuts are defined as a D_S dimensional integral rather than 4-dimensional, when using a quadruple cut as part of generalized unitarity, one no longer finds that four cuts is the maximum number possible without overconstraining the loop momentum. The cut integral still has $D_S - 4$ unconstrained orders of loop momentum. Thus for D_S dimensional Unitarity one is required to consider also 5-particle, "pentuple" cuts, although no higher than this due to the constraints arising on the possible dependence on the $(D-4)$-dimensional parts of the loop momentum, as a result of the external momenta and spins being strictly 4-dimensional. This is as expected from Passarino-Veltman reduction, as discussed in Chapter 2: Reducing a tensor integral to a sum of at most scalar pentagon integrals can be done in general, but expressing the pentagon integrals as a linear combination of scalar boxes is specialised to 4-dimensions.

The actual algorithm used to obtain the coefficients of the scalar integrals is a variant on the OPP method developed by Ellis, Giele and Kunszt [45]. One begins by using an appropriate parameterisation of the loop momentum to extract the coefficients of the pentagon integrals from a system of linear equations satisfying the cut constraints, then with these results proceeding iteratively to extract the box coefficients (including both purely 4-dimensional coefficients and coefficients with a dependence on $\tilde{l}$, where $\tilde{l}$ is defined from the D-dimensional loop momentum by the relation $l^2 = \bar{l}^2 + \tilde{l}^2$, with $\bar{l}$ being the 4-dimensional part of the loop momentum), then the triangle, bubble and finally tadpole contributions. The final result is a decomposition very similar to the OPP integrand reduction, with the difference of having a larger integral basis, containing both pentagon integrals and integrals dependent upon $\tilde{l}$. The full integrand reduction is given by,

$$\begin{aligned} A_N^{(D)} = & \sum_{i_1 \leq i_2 \leq i_3 \leq i_4 \leq i_5}^{N} e^{(0)}_{i_1 i_2 i_3 i_4 i_5} I^{(D)}_{i_1 i_2 i_3 i_4 i_5} \\ & + \sum_{i_1 \leq i_2 \leq i_3 \leq i_3}^{N} \left(d^{(0)}_{i_1 i_2 i_3 i_4} I^{(D)}_{i_1 i_2 i_3 i_4} - \frac{D-4}{2} d^{(2)}_{i_1 i_2 i_3 i_4} I^{D+2}_{i_1 i_2 i_3 i_4} + \frac{(D-4)(D-2)}{4} d^{(4)}_{i_1 i_2 i_3 i_4} I^{(D+4)}_{i_1 i_2 i_3 i_4} \right) \\ & + \sum_{i_1 \leq i_2 \leq i_3}^{N} \left(c^{(0)}_{i_1 i_2 i_3} I^{(D)}_{i_1 i_2 i_3} - \frac{(D-4)}{2} c^{(9)}_{i_1 i_2 i_3} I^{(D+4)}_{i_1 i_2 i_3} \right) \\ & + \sum_{i_1 \leq i_2}^{N} \left(b^{(0)}_{i_1 i_2} I^{(D)}_{i_1 i_2} - \frac{(D-4)}{2} b^{(9)}_{i_1 i_2} I^{(D+2)}_{i_1 i_2} \right) + \sum_{i_1}^{N} a^{(0)}_{i_1} I^{(D)}_{i_1} \,. \end{aligned} \tag{1.3.12}$$

In order to compute physical scattering amplitudes this reduction must be considered in the limit $D \to 4 - 2\epsilon$. In this case one obtains two distinct types of terms: A UV-divergent part arising from the D-dimensional integrals analogous to the cut-constructible part of 4-dimensional Unitarity formalisms, and a part finite in the limit $\epsilon \to 0$ arising from the $D+2$ and $D+4$ dimensional integrals, which yields the rational term. This is one of the key advantages of D-dimensional Unitarity over other formalisms: One can obtain the entire loop amplitude including purely rational pieces using Unitarity, by exploiting the fact that the rational terms are cut-constructible in general in D-dimensions, even though they contain no branch cuts in the special case $D = 4$.

The D-dimensional Unitarity method has been extended to particles with external fermions and massive particles [46], has been implemented numerically as the "Rocket" package [47], and has been applied to compute the process $pp \rightarrow W + 3jet$ [48] as well as being used in the construction of an NLO Monte Carlo event generator [49].

Chapter 2

Background and Outline of the Unitarity Method

2.1 Yang-Mills Theory

In addition to the inherent challenges present in performing accurate calculations of non-trivial loop amplitudes, the computational difficulty of such amplitudes is severely compounded by the various additional problems encountered as a result of working in the typically complicated theories of contemporary phenomenological interest. However, in order to make significant progress in computing experimentally interesting processes, NLO calculations in such non-trivial theories are necessary. The majority of the demand for modern NLO calculations arises from Yang-Mills theory and Quantum Gravity. Although these two theories present significantly different computational challenges, it is still often the case that techniques developed in one may find application in the other; for instance, progress in adapting the BCFW recursion technique developed for calculations in Yang-Mills theory [70, 71] to compute certain classes of amplitudes in $\mathcal{N} = 8$ Supergravity [51], or the possibility that the absence of all scalar integrals below box integrals in $\mathcal{N} = 4$ SYM may suggest a similar property in $\mathcal{N} = 8$ SUGRA, the 'no-triangles' hypothesis [52, 53]; however, this thesis will consider only computations in Yang-Mills theory.

A Yang-Mills theory is an $SU(N)$ gauge theory with N massless fermions and N^2-1 massless vector bosons in the adjoint representation. It was originally developed in the 1950s [54] in an attempt to describe the strong nuclear force; the underlying principle of a non-Abelian gauge theory later proved to be the correct formalism to describe the strong interaction sector of the Standard Model in the form of QCD, an $SU(3)$ gauge theory coupled to fermions in the fundamental representation. As such the study of processes in Yang-Mills theory is phenomenologically important to computing QCD process and backgrounds at the LHC and other colliders, and as a result has been a major focus of theoretical research in recent years.

A pure Yang-Mills theory has the Lagrangian

$$\mathcal{L}_{gauge} = -\frac{1}{4}(F^{\alpha}_{\mu\nu})^2 = -\frac{1}{2}Tr[(F^k_{\mu\nu}T^k)^2], \tag{2.1.1}$$

where the $F^{\mu\nu\alpha}$ is defined in terms of the covariant derivative,

$$F^k_{\mu\nu}T^k = \frac{i}{g}[D_\mu, D_\nu], \tag{2.1.2}$$

$$D_\mu = \partial_\mu - ig(T^k)A^k_\mu, \tag{2.1.3}$$

which leaves the Lagrangian invariant under a local $SU(N)$ gauge transformation.

A typical scattering amplitude in Yang-Mills theory can be computed using Feynman diagrams. In general the resulting expressions will have some complicated dependence both upon the kinematic quantities in the problem and upon the colour structure of the amplitude. This typically results in a lengthy calculation requiring one to keep track of both properties simultaneously. This cumbersome calculation however can be greatly simplified using the colour-sum approach, a good review of which is given by [55]. This technique allows one to essentially separate the problem into two distinct pieces, one of calculating partial amplitudes dependent only upon the kinematic quantities, and a second consisting of a simple trace over group theory colour generators.

In $SU(N)$ QCD, with fermions in the fundamental representation and gluons in the adjoint representation, the purely colour structure parts of the Feynman rules (i.e. neglecting all purely kinematic parts) each give a specific contribution, as listed in the following.

- Quark-quark-gluon vertex: Yields the traceless, Hermitian group generator matrix, $(T^a)^i_j$.
- Gluon 3-vertex: Gives a factor f^{abc}, the colour structure factor, defined by the commutator of the generators,

$$[T^a, T^b] = \frac{i}{\sqrt{2}} f^{abc} T^c\,, \tag{2.1.4}$$

  where the generators are normalized as

$$Tr[T^a T^b] = \delta^{ab}\,,$$

  in order to avoid proliferation of factors of $\sqrt{2}$ in intermediate expressions in accordance with the convention used by Mangano [56].
- Gluon 4-vertex: Gives a pair of contracted colour structure factors, $f^{abe} f^{cde}$.
- Gluon propagator: Yields a δ-function contracting the adjoint colour indices of the vertices connected, δ_{ab}.
- Quark propagator: Yields a δ-function contracting the colour indices of the vertices connected, δ^i_j.

The aim of the approach is to rewrite all of these colour structures in a form where a single overall colour factor can be extracted for a given kinematic configuration. This can be done firstly using the formula

$$f^{abc} = \frac{-i}{\sqrt{2}}(Tr[T^a T^b T^c] - Tr[T^a T^c T^b]), \tag{2.1.5}$$

which can be obtained by contracting both sides of the structure factor definition equation (2.1.4) by T^c, and has the effect of replacing a gluon 3- or 4- vertex with a single or pair of qqg vertices respectively (purely in terms of the colour structure, not with respect to kinematic considerations). Secondly one can Fierz rearrange pairs of uncontracted generator matrices to obtain the formula,

$$T^{ai_1}_{j_1} T^{bi_2}_{j_2} = \delta^{i_1}_{j_2}\delta^{i_2}_{j_1} - \frac{1}{N_c}\delta^{i_1}_{j_1}\delta^{i_2}_{j_2}. \tag{2.1.6}$$

One can apply these formulae to a general all-gluon tree amplitude. Such an amplitude will be constructed from Feynman diagrams consisting of strings of gluon 3- and 4- vertices, i.e. a contracted string of structure factors. Eliminating these by applying equation (2.1.5) reduces the Feynman graph to a product of many traces of generators; applying Fierz rearrangement to these can reduce each term to a single trace. Since one notes from equation (2.1.5) that terms which differ only by the permutation of a single pair of colour indices in the trace contain a relative minus sign, this implies that the second term in equation (2.1.6) will in general cancel against the identical term arising from the trace with a single pair of colour indices permuted.

The first term in equation (2.1.6) meanwhile will survive, being unique to the specific permutation of the T^a's. These unique terms will thus leave us with a sum of single traces, with one such trace for each permutation of the colour indices which can arise from equation (2.1.5).

The full process for the general n-gluon amplitude thus yields the closed form [56]

$$\mathcal{A}^{tree}_n(p_i, h_i, a_i) = g^{n-2} \sum_{\sigma \in S_n \setminus Z_n} Tr(T^{a_{\sigma(1)}} T^{a_{\sigma(2)}} \dots T^{a_{\sigma(n)}}) A^{tree}_n(\sigma(1^h), \dots, \sigma(n^h)). \tag{2.1.7}$$

In the above expression, p_i, h_i and a_i are the set of external momenta, helicities and colour indices in the amplitude, respectively. The set S_n denotes the full set of possible permutations of the n external momenta, and the set Z_n denotes the set of all permutations which can be obtained by cyclic permutation of another term. Thus the set $S_n \setminus Z_n$ simply denotes the set of all permutations of the n external legs which are not cyclically related. We have thus obtained an expression which separates the problem into two parts; the full amplitude, $\mathcal{A}_n^{tree}$ can be computed by a sum of terms each consisting of a trace dependent only upon the colour generator matrices, with no kinematic information, multiplied by a partial amplitude depending purely upon the kinematic information, i.e. the momenta and helicities of the external particles. The task of computing the full amplitude is thus separated into two distinct parts, both simpler than the full calculation. The price is that we must calculate each of the possible non-cyclic helicity permutations individually since the partial amplitudes are not related by crossing symmetry unlike the full amplitude.

However, some partial amplitudes can be obtained from others by symmetry rather than being computed directly. For example, partial amplitudes with the same ordering of helicities but different momenta can be obtained by simple relabeling, e.g.

$$A_6^{tree}(1^-,2^+,3^-,4^+,5^-,6^+) \equiv A_6^{tree}(3^-,4^+,5^-,6^+,1^-,2^+) \equiv A_6^{tree}(5^-,6^+,1^-,2^+,3^-,4^+)\,. \tag{2.1.8}$$

Meanwhile others can be related by parity,

$$A_6^{tree}(1^-,2^-,3^+,4^+,5^+,6^+) \equiv \overline{A_6^{tree}(1^+,2^+,3^-,4^-,5^-,6^-)}\,. \tag{2.1.9}$$

Thus to compute the amplitude $\mathcal{A}_6^{tree}(\{k_1,k_2,k_3,k_4,k_5,k_6\},\{-,-,-,+,+,+\},\{a_i\})$, one requires only the partial amplitudes which correspond to a cyclically unique helicity configuration.

A similar analysis can be applied at loop level. The main difference, in the general all-gluon, 1-loop case, occurs at the stage where we apply equation (2.1.6) to sew together the individual traces yielded by breaking up the structure factors using equation (2.1.5). We still obtain the leading-order in N_c unique terms which lead to the single trace terms, however the presence of internal colour indices causes the possibility of the second term in equation (2.1.6) yielding a non vanishing term. Since the second term in equation (2.1.6) is equivalent (in terms of colour structure) to a pair of gluons not interacting, this will result in the presence of two multiplied colour traces whose spinor indices are not contracted. The full amplitude thus reduces to the form [57]

$$\mathcal{A}_n^{loop}(\{p_i\},\{h_i\},\{a_i\}) = g^n \sum_{\sigma\in S_n\setminus Z_n} N_c Tr(T^{a_{\sigma(1)}}\cdots T^{a_{\sigma(n)}}) A_{n;1}^{loop}(\sigma(1^{h_1}),\ldots,\sigma(n^{h_n}))$$
$$+g^n \sum_{c=2}^{\lfloor\frac{n}{2}\rfloor+1} \sum_{\sigma\in S_n\setminus Z_{n;c}} Tr(T^{a_{\sigma(1)}}\cdots T^{a_{\sigma(c-1)}}) Tr(T^{a_{\sigma(c)}}\cdots T^{a_{\sigma(n)}}) A_{n;c}^{loop}(\sigma(1^{h_1}),\ldots,\sigma(n^{h_n}))\,. \tag{2.1.10}$$

The first sum in the above expression is very similar to the sum in the tree level case; it arises from the single-trace terms which are leading order in N_c. The major difference is the presence of the second sum, arising from the double-trace parts. This contains the set $Z_{n;c}$, which is defined as the set of all permutations of elements of S_n which preserve the *double* trace, thus the set $S_n \setminus Z_n$ is defined as the set of all permutations with unique double traces. The second sum is necessary as we must sum over both all possible non-cyclic permutations of a given division of the generator matrices into a pair of traces, and over all possible divisions between the two traces. The end point of the sum ensures this sum covers all valid divisions for both odd and even numbers of external gluons; specifically, the notation $\lfloor\frac{n}{2}\rfloor$ denotes the largest integer smaller than or equal to $\frac{n}{2}$.

The $A_{n;c}$ are specifically those partial amplitudes left by the remaining kinematic information left multiplying the double traces. In the all-gluon case these are simply given by sums of permutations of the single-trace partial amplitudes $A_{n;1}$ (since unlike the single traces, the double traces do not arise from a unique permutation of external momenta). The whole problem of computing scattering amplitudes is thus largely reduced to the problem of computing the purely kinematic partial amplitudes, and as such any reference to "amplitudes" hereafter in the thesis is a shorthand reference to a *partial* amplitude unless explicitly stated otherwise. It is important to note that although the division of the amplitude into a sum of pure colour traces multiplied by partial amplitudes has been done here using a Feynman diagram analysis, the partial amplitudes themselves are self-contained and can be computed using any applicable method for solving scattering amplitudes, such as BCF-recursion for the tree amplitudes or Unitarity for the loop case.

When considering all-gluon amplitudes, another technique for dividing the single formidable problem of computing loop amplitudes into a sum of objects which are simpler to calculate presents itself from considerations of supersymmetric theories.

2.2 Supersymmetry

It was shown in 1967 by Coleman and Mandula [58] that the known Poincare symmetry, discrete CPT symmetries and internal symmetries are the only possible bosonic symmetries (i.e. symmetries defined by a commutation relation) that a field theory can possess. As such the only possibility to extend the symmetry comes by introducing generators obeying anticommutation relations. Such symmetries are generally termed Supersymmetries, and a number of reviews exist, for example [59]. The generators of this extended symmetry are Q_α and $\bar{Q}_{\dot{\alpha}}$ (for $\mathcal{N} = 1$ supersymmetry), which have the effect of lowering or raising the spin of the particle state they operate on by $\frac{1}{2}$ respectively and obey the commutation relations

$$\begin{aligned} \{Q_\alpha, \bar{Q}_{\dot{\beta}}\} =& 2\sigma^\mu_{\alpha\dot{\beta}} P_\mu \,, \\ \{Q_\alpha, Q_\beta\} =& \{\bar{Q}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\} = 0 \,, \end{aligned} \tag{2.2.1}$$

where P_μ are the generators of translations in the Poincaré group. One can then construct ladder operators in order to find the massless representations by starting from a choice of the maximal helicity state $|\lambda_0\rangle$ such that $a^\dagger|\lambda_0\rangle = 0$ and successively applying a to construct all possible states, where

$$a \propto Q, \quad a^\dagger \propto \bar{Q} \,. \tag{2.2.2}$$

For $\mathcal{N} = 1$ supersymmetry, there are two possible massless representations. If we choose $\lambda_0 = \frac{1}{2}$, we obtain a fermion and a complex scalar, with CPT invariance implying that the conjugate states must exist; this is known as the Chiral supermultiplet. The full particle content is thus

λ	$-\frac{1}{2}$	0	$\frac{1}{2}$
# of states	1	2	1

Figure 2.1: Particle content of the $\mathcal{N} = 1$ Chiral supermultiplet

The other representation (the Gauge or Vector supermultiplet) is obtained by choosing $\lambda_0 = 1$, in which case we obtain a vector boson and a fermion, together with their conjugates.

One can also consider theories with more than one order of supersymmetry; in particular we can consider $\mathcal{N} = 4$ supersymmetry, which has the anticommutation relations

$$
\begin{aligned}
\{Q^i_\alpha, \bar{Q}^j_{\dot\beta}\} =& 2\sigma^\mu_{\alpha\dot\beta} P_\mu \delta^{ij} \,, \\
\{Q^i_\alpha, Q^j_\beta\} =& \epsilon_{\alpha\beta} Z^{ij} \,, \\
\{\bar{Q}^i_{\dot\alpha}, \bar{Q}^j_{\dot\beta}\} =& \epsilon_{\dot\alpha\dot\beta} (Z^{ij})^* \,,
\end{aligned}
\tag{2.2.3}
$$

where the Z^{ij} are central charges which commute with all the other symmetry generators, and $i, j = 1, 2, 3, 4$. In this case we can iterate the application of a^i to the states generated from the maximal helicity state, since $a^i a^j |\lambda_0\rangle \neq 0$ for $i \neq j$. As such the particle content of the CPT self-conjugate representation of $\mathcal{N} = 4$ supersymmetry, found by the choice $|\lambda_0\rangle = 1$ and known as $\mathcal{N} = 4$ Super Yang-Mills, is shown in table 2.2.

λ	1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1
# of states	1	4	6	4	1

Figure 2.2: Particle content of $\mathcal{N} = 4$ Super Yang-Mills

Since no experimental evidence has yet been found for possible realizations of supersymmetry in nature such as the MSSM, we are ultimately primarily interested in computing amplitudes in non-supersymmetric QCD, i.e. those all-gluon amplitudes with either a gluon or a fermion circulating in the loop. However, we can simplify the process of computing these amplitudes by considering all-gluon loops with supersymmetric multiplets circulating in the loop, an analysis first explicitly stated in [73]. For example, from table 2.2 it is clear that a one-loop gluon partial amplitude with an $\mathcal{N} = 4$ Supersymmetric Yang-Mills multiplet circulating in the loop is, due to the definition of the $\mathcal{N} = 4$ SYM particle content, necessarily a sum of gluon, fermion and complex scalar loops,

$$
A_n^{\mathcal{N}=4} = A_n^{[1]} + 4A_n^{[\frac{1}{2}]} + 3A_n^{[0]} \,. \tag{2.2.4}
$$

Similarly an all gluon partial amplitude with an $\mathcal{N} = 1$ chiral multiplet circulating in the loop is a sum of a fermion loop and a complex scalar loop, as can be seen from table 2.2,

$$
A_n^{\mathcal{N}=1_{chiral}} = A_n^{[\frac{1}{2}]} + A_n^{[0]} \,. \tag{2.2.5}
$$

Although these supersymmetric loops are not explicitly present in any known physical amplitude, it is important to note that due to the inherent cancellations between loop amplitudes differing only by SUSY particle types present in Supersymmetric theories, it is actually easier to compute these amplitudes with SUSY multiplet loops than to compute loops with physical particles circulating. It is therefore beneficial to rewrite the gluon and fermion loops of interest for QCD calculations as sums of Supersymmetric amplitudes and a complex scalar loop,

$$\begin{aligned} A_n^{[1]} &= A_n^{\mathcal{N}=4} - 4A_n^{\mathcal{N}=1_{chiral}} + A_n^{[0]}\,, \\ A_n^{[\frac{1}{2}]} &= A_n^{\mathcal{N}=1_{chiral}} - A_n^{[0]}\,. \end{aligned} \tag{2.2.6}$$

Again this naively seems counterproductive since we are replacing a single computation with a sum of objects to be calculated; however the $\mathcal{N}=1$ chiral and in particular the $\mathcal{N}=4$ contributions are much simpler to evaluate than the non-supersymmetric loops; the scalar loop is the most formidable part to solve, but still significantly more straightforward to evaluate due to the lack of spin information circulating in a scalar loop.

2.3 Spinor-Helicity Formalism

In massive 4-dimensional field theories, fermionic particles are described by four-component Dirac spinors. In the case of massless theories however, the structure of the theory simplifies considerably. Ordinarily the Dirac Lagrangian mixes all four components of the Dirac spinor due to the presence of the Dirac gamma matrices,

$$\mathcal{L}_{Dirac} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi\,, \tag{2.3.1}$$

thus if we write the Dirac spinor in the form $\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$, then with the Dirac matrices in the Weyl representation the Dirac Lagrangian becomes

$$\mathcal{L}_{Dirac} = (\psi_R^\dagger, \psi_L^\dagger)\begin{pmatrix} -m & i\delta_\mu\sigma^\mu \\ -i\delta_\mu\sigma^\mu & -m \end{pmatrix}\begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}, \tag{2.3.2}$$

$$\mathcal{L}_{Dirac} = -m\psi_R^\dagger\psi_L + i\psi_R^\dagger\delta_\mu\sigma^\mu\psi_R - i\psi_L^\dagger\delta_\mu\sigma^\mu\psi_L - m\psi_L^\dagger\psi_R\,. \tag{2.3.3}$$

From the above it is clear that the mixing between the positive and negative chirality parts of ψ arises from the mass term; in the case of massless fermions there is no mixing between ψ_L and ψ_R,

$$\mathcal{L}_{Dirac} = i\psi_R^\dagger\delta_\mu\sigma^\mu\psi_R - i\psi_L^\dagger\delta_\mu\sigma^\mu\psi_L\,. \tag{2.3.4}$$

In this case, since the Dirac Lagrangian has effectively decoupled into two independent

Lagrangians dependent on 2-component Weyl spinors, it is not surprising that this decoupling can be made manifest in kinematic quantities appearing in scattering calculations. This scheme is known as the Spinor-Helicity formalism [60]. We can introduce the two-component Weyl spinors λ and $\tilde{\lambda}$ defined such that they obey the inner product,

$$\lambda^i_a \lambda_{i,b} = \epsilon_{ij} \lambda^i_a \lambda^j_b \,, \tag{2.3.5}$$

$$\tilde{\lambda}^{\dot{i}}_a \tilde{\lambda}_{\dot{i},b} = \epsilon_{\dot{i}\dot{j}} \tilde{\lambda}^{\dot{i}}_a \tilde{\lambda}^{\dot{j}}_b \,, \tag{2.3.6}$$

where ϵ_{ij} is the 2-dimensional Levi-Cevita tensor, defined such that $\epsilon_{11} = \epsilon_{22} = 0$, $\epsilon_{12} = 1$, $\epsilon_{21} = -1$, which means the inner product has the explicit form

$$\epsilon_{ij} \lambda^i_a \lambda^j_b = \lambda^1_a \lambda^2_b - \lambda^2_a \lambda^1_b \,, \tag{2.3.7}$$

$$\epsilon_{\dot{i}\dot{j}} \tilde{\lambda}^{\dot{i}}_a \tilde{\lambda}^{\dot{j}}_b = \tilde{\lambda}^1_a \tilde{\lambda}^2_b - \tilde{\lambda}^2_a \tilde{\lambda}^1_b \,. \tag{2.3.8}$$

Since these inner products of the left and right handed spinors will appear frequently (indeed most of the kinematics of scattering amplitudes will ultimately be written in terms of them) these specific products are hereafter referred to as *spinor* products, and denoted by the shorthand,

$$\begin{aligned} \langle ab \rangle &= \epsilon_{ij} \lambda^i_a \lambda^j_b \,, \\ [ab] &= \epsilon_{\dot{i}\dot{j}} \tilde{\lambda}^{\dot{i}}_a \tilde{\lambda}^{\dot{j}}_b \,. \end{aligned} \tag{2.3.9}$$

We can identify the left and right-handed Weyl spinors with the left and right-handed projections of the Dirac spinors. Specifically,

$$\begin{aligned} &\overline{u(k_i)}\tfrac{1}{2}(1-\gamma_5) = \overline{u_+(k_i)} \equiv [i| \quad \tfrac{1}{2}(1-\gamma_5)u(k_i) = u_-(k_i) = |i] \,, \\ &\overline{u(k_i)}\tfrac{1}{2}(1+\gamma_5) = \overline{u_-(k_i)} \equiv \langle i| \quad \tfrac{1}{2}(1+\gamma_5)u(k_i) = u_+(k_i) = |i\rangle \,. \end{aligned} \tag{2.3.10}$$

An almost identical set of relations holds for the outgoing fermion spinors $v(k)$,

$$\begin{aligned} &\overline{v(k_i)}\tfrac{1}{2}(1+\gamma_5) = \overline{v_+(k_i)} \equiv \langle i| \quad \tfrac{1}{2}(1+\gamma_5)v(k_i) = v_-(k_i) = |i\rangle \,, \\ &\overline{v(k_i)}\tfrac{1}{2}(1-\gamma_5) = \overline{v_-(k_i)} \equiv [i| \quad \tfrac{1}{2}(1-\gamma_5)v(k_i) = v_+(k_i) = |i] \,. \end{aligned} \tag{2.3.11}$$

We therefore do not distinguish between incoming and outgoing particles within the formalism, describing both types with angle and square spinors.

The spinors can be related to the 4-momentum via the matrix,

$$p_{a\dot{a}} = \lambda_a \tilde{\lambda}_{\dot{a}} \,. \tag{2.3.12}$$

This can be related back to the Minkowski 4-momentum p^μ by

$$p^\mu = \sigma^\mu_{a\dot{a}} p^{a\dot{a}} . \tag{2.3.13}$$

We can also relate the spinor products to the Lorentz invariant product of two momenta,

$$s_{ab} = 2p_a \cdot p_b = \langle ab\rangle [ba] . \tag{2.3.14}$$

These Weyl spinors have a number of useful properties. One of the most notable is their antisymmetry under reordering,

$$\begin{aligned} \langle ab\rangle &= -\langle ba\rangle , \\ [ab] &= -[ba] , \end{aligned} \tag{2.3.15}$$

which is easy to verify from the definition of the spinor product,

$$\begin{aligned} \langle ab\rangle &\equiv \epsilon_{ij}\lambda^i_a\lambda^j_b , \\ &= -\epsilon_{ji}\lambda^i_a\lambda^j_b , \\ &= -\langle ba\rangle . \end{aligned} \tag{2.3.16}$$

This also implies the property,

$$\langle aa\rangle = [aa] = 0 . \tag{2.3.17}$$

A less trivial identity involving spinors, but one which will be used repeatedly throughout this thesis, is the Schouten identity, which applies to situations where spinors are in a cyclic sum in a pair of spinor products,

$$\langle ab\rangle\langle cd\rangle + \langle ac\rangle\langle db\rangle + \langle ad\rangle\langle bc\rangle = 0 . \tag{2.3.18}$$

This can be anticipated due to the definition of the spinor products in terms of the Levi-Cevita tensor. Since this identity relates four different momenta to each other, it is unsurprising that it becomes applicable very often in a calculation of an amplitude with many external particles.

We can also contract alternating spinor products of the same momentum thus, referred to as "spinor strings", e.g.

$$[ab]\langle bc\rangle = [a|\not{b}|c\rangle , \tag{2.3.19}$$

$$\langle ab\rangle[bc]\langle cd\rangle = \langle ab\rangle[b|\not{c}|d\rangle = \langle a|\not{b}\not{c}|d\rangle , \tag{2.3.20}$$

$$\sum_i \langle aP_i\rangle[P_i b] = [a|\not{P}|b\rangle . \tag{2.3.21}$$

This is largely just a convenient shorthand, although it is worth noting how in the last line it allowed a "massive" momentum P (massive in the sense that $P^2 \neq 0$) consisting of a linear combination of on-shell momenta to be incorporated smoothly into the spinor helicity framework. In addition it allows us to prove an alternative form of the Schouten identity which applies to such spinor strings of massive momenta and which appears frequently in canonical form derivations,

$$2P.Q\langle ab\rangle = \langle a|PQ|b\rangle + \langle a|QP|b\rangle \,. \tag{2.3.22}$$

This can be shown by considering the explicit sums of P and Q,

$$2P.Q\langle ab\rangle = \sum_i \sum_j [P_i Q_j]\langle Q_j P_i\rangle\langle ab\rangle \,. \tag{2.3.23}$$

Applying the original Schouten identity to the angle products yields the desired identity,

$$\begin{aligned}
\sum_i \sum_j [P_i Q_j]\langle Q_j P_i\rangle\langle ab\rangle =& \sum_i \sum_j [P_i Q_j](\langle aQ_j\rangle\langle bP_i\rangle + \langle aP_i\rangle\langle Q_j b\rangle) \,, \\
=& \sum_i \sum_j \langle aQ_j\rangle[Q_j P_i]\langle P_i b\rangle + \langle aP_i\rangle[P_i Q_j]\langle Q_j b\rangle \,, \\
=& \langle a|QP|b\rangle + \langle a|PQ|b\rangle \,.
\end{aligned} \tag{2.3.24}$$

Finally, there is a special case of this form of the Schouten identity which appears particularly often, namely the case $P = Q$, where we obtain,

$$P^2\langle ab\rangle = \langle a|PP|b\rangle \,. \tag{2.3.25}$$

In order to rewrite a Feynman diagram fully in terms of the spinor helicity formalism we must also have a spinor helicity form of the polarisation tensors for the gluons [61],

$$\epsilon^+_\mu(p,\eta) = \frac{[p|\gamma_\mu|\eta\rangle}{\sqrt{2}\langle\eta p\rangle} \,, \tag{2.3.26}$$

$$\epsilon^-_\mu(p,\eta) = -\frac{[\eta|\gamma_\mu|p\rangle}{\sqrt{2}[\eta p]} \,, \tag{2.3.27}$$

where the momentum η is an arbitrary reference momentum. The presence of this degree of freedom is a consequence of the gauge freedom of the vector boson; this can be observed by noting that η carries zero overall spinor and momentum weight in the above expressions.

The primary advantage of using the spinor-helicity formalism is that although in principle one must do more calculations by calculating helicity configurations individually rather than summing over all states using the completeness relations, in practice a very large number of possible configurations either vanish or else have very simple expressions when written in spinor-helicity form. In particular, at tree level we have the results that in general, all-gluon amplitudes with all helicities identical, and all-gluon amplitudes with only one particle of differing helicity to the others, vanish for any number of external particles,

$$\begin{aligned} A_n^{tree}(1^-, 2^-, \ldots, n^-) =& A_n^{tree}(1^+, 2^+, \ldots, n^+) = 0\,, \\ A_n^{tree}(1^-, 2^-, \ldots, a^+, \ldots, n^-) =& A_n^{tree}(1^+, 2^+, \ldots, a^-, \ldots, n^+) = 0\,, \end{aligned} \tag{2.3.28}$$

while for the case with two particles of negative helicity and the remainder positive we have the formula postulated by Parke and Taylor [62] and proven by Berends and Giele [63], expressing the amplitude as a single term, for an arbitrary number of external particles

$$A_n^{tree}(1^+, \ldots, a^-, \ldots, b^-, \ldots, n^+) = \frac{\langle ab\rangle^4}{\prod_{i=1}^{n}\langle i, i+1\rangle\langle n1\rangle}\,. \tag{2.3.29}$$

This helicity configuration is commonly known as the Maximally Helicity Violating or MHV configuration. Together with the equivalent formula for the opposite helicity case (known as the $\overline{\text{MHV}}$ or "googly"),

$$A_n^{tree}(1^-, \ldots, a^+, \ldots, b^+, \ldots, n^-) = \frac{[ab]^4}{\prod_{i=1}^{n}[i, i+1][n1]}\,. \tag{2.3.30}$$

This gives all non-vanishing gluonic tree amplitudes up to 5-point. It will become clear in Chapters 4 and 5 that this fact is extremely helpful, since it implies Unitarity cuts in those amplitudes never contain more than one NMHV tree amplitude. Meanwhile many other tree amplitudes, such as those with all external gluons except for a pair of external gluinos or complex scalars, can be obtained from the MHV amplitudes via Supersymmetric Ward identities [64].

2.4 Integral Reduction

A general n-point tensor integral appearing in a loop amplitude in the dimensional regularization scheme has the form,

$$\frac{(2\pi\mu)^{4-d}}{i\pi^2}\int d^d l \frac{l_{\mu_1}\cdots l_{\mu_r}}{D_0\cdots D_{n-1}}\,, \tag{2.4.1}$$

where the D_i denote the loop propagators, $D_i = (l+p_i)^2 - m_i^2$. Evaluating such an integral is highly non-trivial, especially considering the large number of possible distinct integrals which can be constructed for any reasonably high value of m. As a result the only practical way to approach a non-trivial loop calculation is to attempt to write the necessary integrals in terms of a decomposition of simpler functions. A systematic prescription for writing general tensor integrals in terms of a decomposition of simpler scalar integrals multiplied by l-independent coefficients was famously developed by Passarino and Veltman in 1979 [1], capitalising upon the earlier work by Veltman and 't Hooft in obtaining solutions for the various scalar integrals in the dimensional regularization scheme [65]. The prescription requires that one decompose the tensor integral in terms of some basis of all possible covariant quantities which can be constructed from the parameters of the amplitude (namely the external momenta p_i^μ and the metric $g^{\mu\nu}$) multiplied by unknown scalar coefficients which are to be determined,

$$I_m^{\mu_1\cdots\mu_r} = \sum_{\mathcal{P}(p_i,g)} \mathcal{P}^{\mu_1\cdots\mu_r} I_{\mathcal{P}}\,. \tag{2.4.2}$$

For example, the quadratic triangle has the decomposition,

$$C^{\mu\nu} = g^{\mu\nu}C_{00} + p_1^\mu p_1^\nu C_{11} + (p_1^\mu p_2^\nu + p_2^\mu p_1^\nu)C_{12} + p_2^\mu p_2^\nu C_{22}\,. \tag{2.4.3}$$

One then identifies the unknown coefficients by contracting both sides by the external momenta and the metric, and attempting to split the resulting numerator in the tensor integral into a sum of propagators to cancel against those in the denominator, and loop-momentum-independent terms. For instance, contracting a numerator factor l^μ with an external momentum p_i^μ allows one to expand the resulting product,

$$\begin{aligned} p_i^\mu l_\mu =& \frac{1}{2}(p_i+l)^2 - \frac{1}{2}p^2 - \frac{1}{2}l^2\,, \\ =& \frac{1}{2}((p_i+l)^2 - m_i^2) - \frac{1}{2}(l^2 - m_0^2) - \frac{1}{2}(p^2 + m_0^2 - m_1^2)\,. \end{aligned} \tag{2.4.4}$$

The first two terms are loop propagators which can be cancelled against those present in the denominator to yield integrals of $(n-1)$ point, while the final term yields a constant multiplying an n-point integral of tensor rank $(r-1)$. The result is a system of simultaneous equations which allows one to solve the coefficients in terms of a sum of n-point or lower scalar integrals.

This allows any n-point tensor loop integral of rank r to be written as a decomposition of scalar integrals of n-point or lower. In addition this very powerful result can be extended even further using techniques which allow one to decompose 5-point or higher scalar integrals in terms of 4-point, 3-point, 2-point and rational pieces. This method was originally developed by van Neerven and Vermaseren [66].

The decomposition begins by relating the loop momentum q to the first four external momenta for a pentagon or higher integral using the Schouten identity for the Levi-Civita tensor,

$$\epsilon^{p_1p_2p_3p_4}q^\mu = \epsilon^{\mu p_2p_3p_4}(q.p_1) + \epsilon^{p_1\mu p_3p_4}(q.p_2) + \epsilon^{p_1p_2\mu p_4}(q.p_3) + \epsilon^{p_1p_2p_3\mu}(q.p_4)\,. \tag{2.4.5}$$

For the five point scalar, one can contract this expression with q_μ, which with the shorthand,

$$\begin{aligned} v_1^\mu =&\epsilon^{\mu p_2p_3p_4}, v_2^\mu = \epsilon^{p_1\mu p_3p_4}\,,\\ a =&\epsilon^{p_1p_2p_3p_4}, \bar{a} = \epsilon_{p_1p_2p_3p_4}\,,\\ r_i =&p_i^2 - m_i^2 + m_0^2\,, \end{aligned} \tag{2.4.6}$$

and after dividing by the loop integral propagators and integrating over d^4q, yields

$$\int d^4q\frac{2am_0^2 + (2a + \sum_{i=1}^4 q.v_i)D_0 + \sum_{i=1}^4 r_i(q.v_i)}{D_0D_1D_2D_3D_4} = 0\,. \tag{2.4.7}$$

Manipulating the q-dependent terms to the form $q.p_i$ which can be substituted in terms of propagators to cancel against the denominator and q-independent terms ultimately yields the expression,

$$\int d^4q\frac{\Delta_4(2m_0^2 - \frac{1}{2}w^2) + D_0\Delta_4 - \frac{1}{2}D_0\sum_i v_i.w + \frac{1}{2}\sum_i D_i(v_i.w)}{D_0D_1D_2D_3D_4} = 0\,, \tag{2.4.8}$$

where $w^\mu = \sum_i r_i v_i^\mu$. Thus one can express the 5-point scalar loop as a sum of five 4-point scalar loops,

$$E_{01234}(w^2 - 4\Delta_4 m_0^2) = D_{1234}(2\Delta_4 - \sum_i w.v_i) + D_{0234}v_1.w + D_{0134}v_2.w + D_{0124}v_3.w + D_{0123}v_4.w\,. \tag{2.4.9}$$

For 6 or more points in four dimensions, one has an extra external momentum to contract the Schouten identity with,

$$a(q.p_5) = (v_1.p_5)(q.p_1) + (v_2.p_5)(q.p_2) + (v_3.p_5)(q.p_3) + (v_4.p_5)(q.p_4)\,. \tag{2.4.10}$$

It is simple to substitute for q in the expression, yielding the integral,

$$\int d^4q \frac{\sum_{i=1}^{4} r_i(v_i.p_5) - ar_5 + aD_5 + D_0(\sum_{i=1}^{4}(p_5.v_i) - a) - \sum_{i=1}^{4} D_i(v_i.p_5)}{D_0D_1D_2D_3D_4D_5} = 0\,. \qquad (2.4.11)$$

Thus one can obtain the 6-point scalar loop integral as a sum of six 5-point scalar loops. In addition, since the above derivation did not rely upon the number of denominators, one can generalize the result to an arbitrary n-point scalar integral, provided $n \geq d+2$.

These methods for decomposing complicated loop integrals ultimately imply that it is possible to iteratively decompose the general 4-dimensional loop integral of arbitrary tensor rank and number of propagators in terms of purely scalar box, triangle, bubble, and tadpole integrals plus rational terms in which the decomposition results in an integral containing loop integrals. We thus have the following decomposition for massless theories, in which the tadpoles do not contribute,

$$I(n) = \sum_i \mathcal{D}_i I_i^{(4)} + \sum_j \mathcal{C}_j I_j^{(3)} + \sum_k \mathcal{B}_k I_k^{(2)} + \mathcal{R}\,, \qquad (2.4.12)$$

where the $I^{(4)}$, $I^{(3)}$ and $I^{(2)}$ are the basis of all possible scalar box, triangle and bubble integrals and the $\mathcal{D}_i$, $\mathcal{C}_j$, $\mathcal{B}_k$ and $\mathcal{R}$ are some unknown rational (i.e not containing any branch cuts), loop-momentum-independent terms. Since the scalar loop integrals up to 4-point have known, standard solutions [65, 85], the proof that any n-point integral for $n \geq 5$ has a decomposition of this form reduces the problem of calculating this integral to the problem of identifying these rational coefficients.

In practice it is possible to begin with the desired loop integral and perform the reduction to this integral basis using the methods described above in order to obtain the rational coefficients directly; however, the powerful consequence of this result is that since any desired integral is known to have such a decomposition, one can obtain the rational coefficients by other means in order to *construct* the amplitude from the integral basis as opposed to *reducing* the integral to the basis. This is the approach used in the Unitarity method; the coefficients of all permitted basis integrals are constructed recursively from the tree amplitudes, without knowledge of the specifics of the reduction.

2.5 The Unitarity Method

Although one can in principle apply Passarino-Veltman reduction techniques to a traditional Feynman diagram calculation in order to compute the coefficients of equation (2.4.12), this becomes problematic for many processes of interest due to the extremely rapid proliferation in the number of possible Feynman diagrams as a function of the number of external particles, as illustrated in Chapter 1. Clearly it is impractical for one to perform a Feynman diagram calculation manually in many of the areas of interest, at 5-, 6- or 7-point. Feynman diagram techniques are still being applied successfully to compute relevant amplitudes (e.g. [16, 17]), however all contemporary implementations rely upon computer automation to evaluate the amplitude numerically.

Partly in response to this problem, a great deal of progress has been made over recent years to develop alternative techniques for computing perturbative scattering amplitudes. Some of the techniques developed are of interest primarily for the potential physical insights to be obtained from an alternative formulation of the theory, for example the observation that one can construct amplitudes using MHV diagrams as opposed to Feynman rules [67] and further can reformulate the Yang-Mills Lagrangian in terms of an MHV Lagrangian [68, 69]. However, the most promising avenue of progress in terms of developing techniques to successfully compute new amplitudes appears to be in on-shell techniques relying upon exploiting the pole structure of amplitudes in the complex plane. There are two notable such techniques which have come to prominence, namely BCFW recursion, a technique which has proven very effective for amplitudes containing only poles and no branch cuts, and Unitarity, a method which by contrast exploits the properties of branch cuts of loop integrals and thus can only be applied to amplitudes containing branch cuts. Although the two methods arise from different physical origins, they share some important features and properties. In particular, they both compute a required scattering amplitude by constructing a sum of terms each consisting of a product of simpler amplitudes, and both result in a relatively small number of terms to be calculated compared to a Feynman diagram calculation of the same process.

This differs substantially from the Feynman diagram technique: with the Feynman approach one can compute any amplitude purely from the set of Feynman rules derived from the Lagrangian; in principle one can compute any amplitude without requiring knowledge of the simpler amplitudes in the theory. This can be both an advantage and a disadvantage; recursive methods require the amplitudes up to $(n-1)$ external legs to be known; however, since phenomenological interest in amplitudes very often does build upon earlier work which necessitated the calculation of the previous generation of amplitudes, the simpler amplitudes are very often already known for calculations of interest. Meanwhile the recursive structure allows one to take advantage of the often very simple structure of tree amplitudes in Yang-Mills, particularly the MHV amplitudes, whilst allowing one to compute amplitudes without reference to the Lagrangian.

In addition, the Feynman diagram calculation is purely algebraic at tree level, requiring no integration (of course, at loop level we will still have a non-trivial loop integral to compute). Therefore, in order for these recursive methods to be worthwhile compared to Feynman methods, the reduction in the number of terms to be computed must make up for the increased difficulty of evaluating each term, and for the requirement that all simpler amplitudes needed for the recursive calculation be known. Due to the very poor scaling of the Feynman formalism as the number of external legs increases, on-shell methods show greater promise for computing amplitudes with many external particles; both have a polynomial scaling in number of terms with number of external particles.

The BCFW recursion technique [70, 71] is the only approach which can be applied at tree level. This method relies upon applying some complex shift z to the momenta of two or more of the external legs of the target amplitude. It can then be shown that the amplitude is equal to the closed contour integral over z in either the upper or lower half of the complex plane. Providing that:

1. The shift z is chosen such that the boundary term vanishes.

2. The singularity structure of the desired amplitude consists only of simple or multiple poles, with no branch cuts.

This integral must be equal to the sum of the residues of the poles in z within the contour, these poles arising only from internal z-dependent propagators in a contributing Feynman diagram. The residue of a given pole is thus given by the product of tree amplitudes which are linked by the relevant z-dependent propagator; the relevant term is evaluated by applying the z-shift to each pair of tree amplitudes and applying the relevant choice of z that yields a singularity in the propagator.

The result is a method which allows one to calculate a given scattering amplitude as a sum of products of tree amplitudes of lower number of external particles.

Since relatively few such choices of tree amplitudes can be made, particularly when computing a colour-ordered Yang-Mills partial amplitude in which one is restricted only to momentum channels of adjacent particles, this allows tree amplitudes to be computed very efficiently.

2.5.1 Unitarity

The key to computing coefficients of branch cuts comes from exploiting the Unitarity of the S-matrix to reconstruct the imaginary part of the amplitude from its discontinuities via the Optical Theorem. Cutkosky showed in 1960 [72], that one can compute the discontinuities of any Feynman diagram by first finding all possible ways of "cutting" propagators within the diagram, then for each propagator cut, replacing the propagator with a delta-function effectively sending the cut particle on-shell,

$$\frac{1}{(l^2 - m^2 + i\epsilon)} \to -2\pi i \delta(l^2 - m^2)\,. \tag{2.5.1}$$

When applied to compute the discontinuity of a loop integral, this replacement has the effect of placing an additional constraint on the loop momentum beyond that required by internal momentum conservation, thus effectively reducing the order of the loop momentum integral by one for each propagator cut.

This process becomes particularly useful when one considers the effect of cutting two loop propagators. Diagrammatically this takes the form,

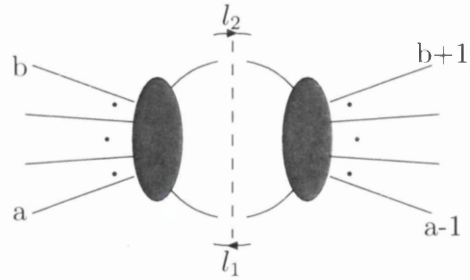


Figure 2.3: A 2-particle cut in the channel $P_{a\cdots b}$

The diagram is thus effectively separated into a product of two tree diagrams, since the legs in each diagram where the propagators have been cut have been replaced by the on-shell loop momentum satisfying $l_1^2 = l_2^2 = 0$. Since this analysis applies equally for the full loop amplitude as well as individual diagrams, we can thus state that this particular discontinuity is given by

$$C_{a,\dots,b} = \frac{i}{2}\int dLIPSA^{tree}(-l_1, a, a+1, \dots, b, l_2) \times A^{tree}(-l_2, b+1, b+2, \dots, a-1, l_1)\,, \quad (2.5.2)$$

where the integration measure $dLIPS$ refers to the Lorentz-invariant phase space measure of the loop integration with the constraints imposed by the cut loop propagators. In this case it would take the explicit form

$$\int dLIPS = \int_{-\infty}^{+\infty} d^d l_1 \delta(l_1^2)\delta\left((l_1 + a + (a+1) + \dots + b)^2\right)\,. \quad (2.5.3)$$

The key to the 4-dimensional Unitarity method as originally developed in 1994 [73, 74] comes when one considers from where in the loop integral, when decomposed into a sum of coefficients multiplying scalar integrals as in equation (2.4.12), one will acquire a non-zero discontinuity in this momentum channel. The parts containing branch cuts are the logarithmic and dilogarithmic terms in the bubble, triangle and box integrals; thus in principle, the coefficients of all the discontinuities of the scalar loop integrals can be computed by constructing such cuts. However, not all members of the integral basis will contain a branch cut singularity in the correct momentum channel to contribute to the particular cut $C_{t_{a,\dots,b}}$; this cut of the full loop amplitude is thus equivalent to the leading discontinuity of all parts of the integral basis which contain a branch cut singularity in this momentum channel. Thus we obtain,

$$C_{a,\dots,b} = \left.\left(\sum_{i\in\mathcal{C}'} a_i I_4^i + \sum_{j\in\mathcal{B}'} b_j I_j^3 + \sum_{k\in\mathcal{A}'} c_k I_k^2\right)\right|_{Disc}\,, \quad (2.5.4)$$

where $\mathcal{C}'$, $\mathcal{B}'$ and $\mathcal{A}'$ are the restricted sets of all integral functions with a cut in the channel $t_{a,\dots,b}$. In particular $\mathcal{A}'$ is given by a single term; the only possible bubble integral with a cut in the channel $t_{a,\dots,b}$ is the integral containing the logarithm $ln(t_{a,\dots,b})$. $\mathcal{B}'$ and $\mathcal{C}'$ however, will generally consist of a sum of terms; any triangle or box integral which can be obtained by inserting an additional loop propagator or two into the bubble integral $I_2^{t_{a,\dots,b}}$. Thus by constructing all possible double cuts, one obtains a set of functions which contain all the possible rational coefficients of the integral basis.

As an example one can enumerate the specific cuts which can in principle appear for a calculation of the cut-constructible parts of a 6-gluon one loop amplitude. For double cuts, one can consider either cuts with two legs on one side and four on the other, referred to in this thesis as s-cuts, or cuts with three legs on either side, referred to as t-cuts. For the s-cuts there are 6 possibile channels: s_{12}, s_{23}, s_{34}, s_{45}, s_{56} and s_{61}. Note that since we are interested in computing only colour-ordered partial amplitudes as defined in equation (2.1.10), one cannot obtain contributions from cuts containing non-adjacent legs, so there can be no cut in the s_{13} channel, for instance. For the t-cuts we have the possibilities t_{123}, t_{234} and t_{345}, with the remaining cuts such as t_{456} being equivalent to these three by momentum conservation. Of course in practice, many of these cuts will not provide a finite contribution since they will contain a tree with a vanishing helicity configuration, such as those in equation (2.3.28).

Thus if one can rewrite the integrand of equation (2.5.2) in such a form that it consists of pure scalar box, triangle and bubble contributions multiplied by l-independent coefficients,

$$\begin{aligned}&\int dLIPS A^{tree}(-l_1, a, a+1, \ldots, b, l_2) \times A^{tree}(-l_2, b+1, b+2, \ldots, a-1, l_1)\,, \\ &\quad = \int dLIPS \left(\sum_i a_i \frac{1}{(l_1 - K_1^i)^2 (l_2 - K_2^i)^2} + \sum_j b_j \frac{1}{(l_1 - K_3^j)^2} + c \right)\,,\end{aligned} \tag{2.5.5}$$

one can identify the coefficients a_i, b_j and c with the coefficients of the integral basis. In particular, it is clear from the appearance of a term of the form

$$c \int dLIPS\,, \tag{2.5.6}$$

that we can identify c as the coefficient of $I_2(P)$ where $P = (a + (a+1) + \ldots + b)$, since by definition

$$\int dLIPS \equiv \int d^d l_1 \frac{1}{l_1^2 (l_1 + P)^2}\bigg|_{disc} = I_2(P)\,. \tag{2.5.7}$$

Thus, using Unitarity cuts to obtain the rational coefficients it is possible to construct all parts of a general 1-loop amplitude in the form of the integral decomposition given in equation (2.4.12), with the exception of the purely rational, $\mathcal{O}(\epsilon)$ parts left over by cancelling all loop propagators during Passarino-Veltman reduction. The parts of the amplitude which can be obtained via Unitarity cuts are generally referred to as the *cut-constructible* parts; the remaining parts, normally referred to as the *rational* (i.e. non cut-containing) parts, cannot be obtained using the 4-dimensional Unitarity approach discussed here and must be found instead using other methods. In cases where amplitudes which are not fully cut-constructible have been computed using a Unitarity implementation, these pieces have been obtained one of three ways:

By using Feynman diagram techniques specialised to obtain the rational terms [75]; by working in a d-dimensional Unitarity formalism as discussed in section 1.3.4; or else to by applying the techniques of BCFW recursion to obtain the rational pieces from the known rational pieces of simpler loop amplitudes, as used in the "Bootstrap" formalism discussed in section 1.3.1.

2.5.2 Generalized Unitarity

Constructing the cut-constructible parts of a 1-loop amplitude using 2-particle Unitarity cuts represents an enormous simplification in the number of terms which must be computed when compared to Feynman techniques; as illustrated in chapter 1, the scaling in the number of Feynman diagrams contributing to a 1-loop amplitude proliferates dramatically as the number of external particles increases. The number of possible 2-particle Unitarity cuts, meanwhile, follows a much more manageable polynomial progression. This improved scaling in the number of terms is counterbalanced somewhat, however, by the fact that to obtain the integral coefficients from the Unitarity cuts, one must either compute the cut integral itself directly as for example in the Direct Parametrization method discussed in section 1.3.1, or else manipulate it into such a form that the parts giving rise to each basis integral can be uniquely identified, the approach used in the original implementation of the Unitarity method [73, 74]. Both approaches result in a non-trivial calculation that is hard to solve in general.

A significant simplification which allows for a more efficient Unitarity implementation is known as *generalized* Unitarity. The method arises from the observation that one need not be restricted only to considering double cuts. One can instead choose to cut three or even four loop propagators in the same way; this has the effect of simplifying the cut integral as each additional cut propagator imposes an additional constraint on the loop momentum, reducing the overall order of the integration by one. The cost of this simplification is that the range of integral functions which have a branch cut in the more restricted choice of momentum channels is lower. Consider, for example, a triple cut as in figure 2.5.2 with momentum channels $t_{a,\ldots,b}$ and $t_{b+1,\ldots,c}$, where $c \neq a-1$. In this case, the cut integral will isolate the coefficient of a single triangle integral with massive corners $t_{a,\ldots,b}$ and $t_{b+1,\ldots,c}$, plus a sum of box integral contributions which can be obtained by inserting a loop propagator into the third corner of the triangle integral.

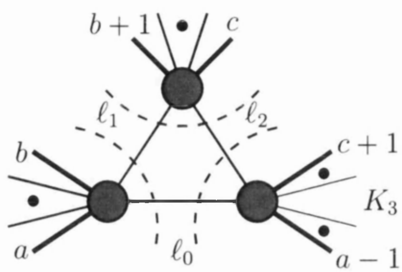


Figure 2.4: The triple cut in the channels $t_{a,\ldots,b}$, $t_{b+1,\ldots,c}$

However, since by momentum conservation it is impossible for a bubble integral to have cuts in both the $t_{a,\ldots,b}$ and $t_{b+1,\ldots,c}$ channels, it is clear that the cost of simplifying the integral using a triple cut is that one cannot obtain any information on the bubble integral coefficients by considering only triple cuts. Triple cuts have been used in Unitarity calculations as far back as 1998 [77], however another significant advance which led to the development of Generalized Unitarity did not come until the development by Britto, Cachazo and Feng in 2005 of techniques for evaluating quadruple Unitarity cuts [78]. Prior to this the development of such cuts had been hindered by the fact that both triple and quadruple cuts generally contain "massless" corners which take the form of a three-particle tree amplitude. For massless particles with real momenta such an amplitude is not well defined on-shell, as can be seen by considering the form of the Parke-Taylor formula for such an amplitude of three gluons, as shown in figure 2.5.8.

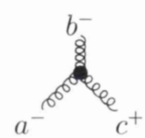


Figure 2.5: The gluon 3-vertex

$$A^{tree}(a^-, b^-, c^+) = \frac{\langle a\, b\rangle^3}{\langle b\, c\rangle \, \langle c\, a\rangle} \,. \tag{2.5.8}$$

If one requires that all three particles be on-shell, then one can show,

$$\langle a\, b\rangle \, [b\, a] = 2a.b = c^2 = 0 \,. \tag{2.5.9}$$

If a and b are real momenta, then $\langle a\, b\rangle$ and $[b\, a]$ must vanish simultaneously since $\lambda_a = \tilde{\lambda}_a{}^*$, and the amplitude (2.5.8) is not well-defined. The key breakthrough came with the observation in 2003 by Witten [76] that one can meaningfully define momenta by an analytical continuation onto the complex plane, such that $\lambda_a \neq \tilde{\lambda}_a^*$. In addition to allowing one to perform the complex integration necessary to BCFW recursion, this fact also allows one to satisfy equation (2.5.9) by only requiring that $[a\,b] = 0$, while $\langle a\,b\rangle \neq 0$. In this case the three point tree (2.5.8) is now a well-defined amplitude, allowing triple and quadruple cuts containing massless corners to be performed in a meaningful manner.

Quadruple cuts take a particularly simple form in 4-dimensional Unitarity. Since the four cuts give rise to four delta-function constraints on the loop momentum, this has the effect of freezing out the loop integral completely and uniquely determining the loop momentum in terms of the external momenta (up to a possible ambiguity in the helicity of the loop particle, which requires one to sum over all possible internal helicity configurations), allowing one to solve the cut purely algebraically. The loop momentum solutions required for the box coefficients considered in this thesis are presented in Appendix C.

As with the triple cut, one can again note that the benefit of simplifying (or in this case eliminating) the integration has come at the cost of reducing the information about the amplitude that the cut contains. Specifically, a quadruple cut directly yields the single box coefficient with branch cuts in the channels specified, and cannot yield information about triangle or bubble coefficients since no triangle or bubble integral could contain cuts in all three channels required.

The approach followed in this thesis is to exploit 4-dimensional Generalized Unitarity to solve each type of integral coefficient using the simplest type of cut possible; all box coefficients are solved algebraically by constructing all possible quadruple cuts allowed by momentum conservation and helicity considerations. The triangle coefficients meanwhile are obtained by considering triple cuts, and manipulating the integrand into such a form that those contributions which give rise only to box coefficients can be identified and discarded, leaving only the single contribution to the triangle. Finally the most difficult type of cut, the double cuts, are used only to obtain the bubble coefficients; any terms appearing in the integrand which can be identified as contributing purely to a triangle or box coefficient can again be discarded. Thus one performs one double, triple or quadruple cut for each non-zero bubble, triangle or box coefficient to be found, respectively.

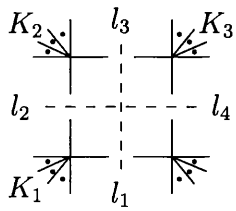
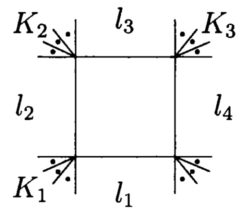
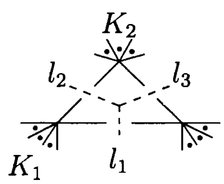
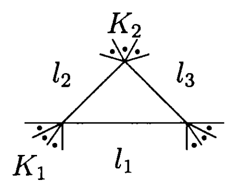
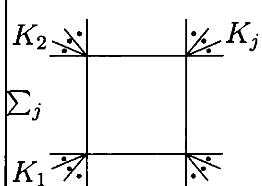
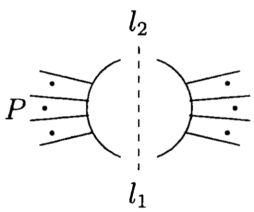
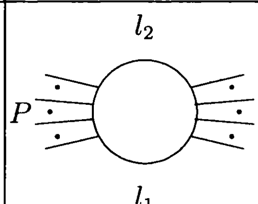
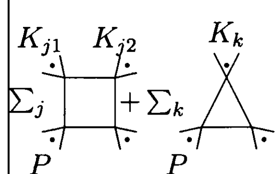
Cut Integral	Target Function	Discarded Terms
 $\int dLIPS A(-l_1, K_1, l_2)$ $\times A(-l_2, K_2, l_3) A(-l_3, K_3, l_4)$ $\times A(-l_4, -K_1 - K_2 - K_3, l_1)$	 $D_i I_4(K_1, K_2, K_3)$	None
 $\int dLIPS A(-l_1, K_1, l_2) A(-l_2, K_2, l_3)$ $\times A(-l_3, -K_1 - K_2, l_1)$	 $C_i I_3(K_1, K_2)$	 $\sum_j D_j I_4(K_1, K_2, K_j)$
 $\int dLIPS A(-l_1, P, l_2) A(-l_2, -P, l_1)$	 $B_i I_2(P)$	 $\sum_j D_j + \sum_k C_k$

Figure 2.6: Illustration of cutting principle, showing information retained and discarded from each cut

Chapter 3

The Canonical Basis Method

3.1 Underlying Principle

As discussed in Chapter 2, the Unitarity method allows the coefficients of the loop integral basis of a one-loop amplitude to be calculated by evaluating the tree amplitudes given by cutting the loop propagators in the momentum channels specified by the branch cuts of the specific integral function. Since cutting propagators has the effect of reducing the power of the loop integral by one per propagator cut, this effectively converts the problem into one of evaluating a number of simpler integrals as opposed to a single difficult loop integral. As mentioned, there is some ambiguity both in the choice of the target integral basis (one can choose to either use d-dimensional or 4-dimensional cuts, with the former allowing the entire amplitude to be calculated using cuts in the massless case, and the latter significantly simplifying the evaluation of the cuts at the cost of requiring some parts of the amplitude to be evaluated by methods other than unitarity), and in the choice of which coefficients will be extracted from which cuts (in principle the full set of double cuts contain all the information to extract all cut-constructible terms, but one can choose to instead extract some parts, namely the triangle and box coefficients, from simpler triple and quadruple cuts).

However, the remaining cut integrals are still non-trivial to evaluate, which considering the large number of cuts required for amplitudes with many external particles, and the increasing size and complexity integrals once NMHV trees begin appearing in cuts, presents a significant practical obstacle to further analytic evaluation of loop amplitudes. As discussed in chapter 1 various techniques have been developed for evaluating the cut integrals both numerically and analytically, but due to the difficulty of in particular the double cut integrals evaluating each integral by hand would result in an impractically large investment of time to calculate any useful amplitude. As such most progress in evaluating interesting phenomenological processes via Unitarity has shifted in the direction of automated numerical evaluation in recent years; an approach which is quite sufficient for experimental purposes. This lack of techniques producing relatively compact analytic expressions for loop amplitude coefficients does however represent a potential obstacle to future progress in loop integral computations; much of the modern approach to perturbative calculations in general, and the Unitarity method in particular, is inherently recursive in structure: rather than computing amplitudes directly from the Feynman rules derived from the Lagrangian, amplitudes are evaluated entirely by examining their analytic pole structure, allowing them to be constructed from simpler, previously evaluated amplitudes. Thus, closed, analytic expressions are necessary as a starting point to calculating further, more complicated amplitudes either analytically or numerically.

The key to the canonical basis approach is to recognise firstly that the most difficult part of the calculation in most Unitarity implementations is the cut integration itself; and secondly, that due to the structure of the cuts being constructed from the same set of tree amplitudes and due to the many ways in which a cut integrand can be manipulated algebraically using the various identities of momentum conservation and the spinor-helicity formalism for massless particles, often many of the structures appearing in individual cuts can be rewritten in such a way as to be equivalent to structures appearing in other cuts; in essence, one observes that the same integral appears in multiple cuts. The Canonical Basis approach therefore splits the process of computing an amplitude into two phases:

1) One postulates some minimal canonical basis of distinct cut integrals to use to construct all required cuts in the desired amplitude. These cut integrals are solved individually by any convenient method to extract a closed, rational, analytic expression for the canonical form. Since the basis has relatively few elements, it is worthwhile to obtain as compact an expression for each form as possible, in particular eliminating unphysical square roots of Gram determinants which appear in the integral solutions derived, for example, by the Fermionic integration technique [37]. Such irrational terms are an undesirable feature since a necessary precondition to use an amplitude in an ingredient for on-shell recursion is that the amplitude is explicitly a rational function of the complex momentum shift.

2) Once a candidate canonical basis is complete, the full set of cuts appearing in the desired amplitude are examined and an attempt is made to manipulate each individual cut into a form composed only of canonical forms appearing in the solved canonical basis, whereupon the solution to the integral can be simply substituted to evaluate the cut. Any cuts which cannot be rewritten in terms of the known basis are identified, and one repeats the process, performing the integration of those new forms which must be added to the canonical basis and applying the result to the relevant cut. These two steps are thus iterated until all the required cuts have been evaluated.

As is hopefully apparent, the main benefit of this approach is to effectively reduce the total amount of integration to be performed, at the cost of introducing more algebra to rewrite initially dissimilar cuts into equivalent forms. Provided that the size of the canonical basis is small relative to the total number of cuts required, and that the extra algebra requires less time to perform than that saved by not integrating term-by-term, this approach thus greatly simplifies the process of performing Unitarity calculations by hand to the point of rendering analytic solutions of unknown loop amplitudes with many external particles practicable.

A further advantage presents itself due to the generality of the canonical basis and the principle of repeated application to similar cuts: since individual canonical forms are as general as possible, and not specific to a given particle content, helicity configuration or number of external legs, it is often the case that part or all of a canonical basis derived for one amplitude can be applied to the cuts of another amplitude, thereby potentially considerably simplifying the calculation of that amplitude. This will be seen to be the case in the applications discussed in chapters 4 and 5.

3.2 Integrand Identities and Nomenclature

As mentioned, there are a potentially large number of loop momentum-dependent structures which can appear in the double cuts of a Yang-Mills amplitude. In order to make the canonical basis approach worthwhile we must be able to manipulate various superficially different structures into the same form. In the applications considered here one can identify three basic structures which can appear in a double cut canonical form and which all required cuts can be expressed in terms of.

Firstly, a double cut integrand may contain a propagator in the denominator dependent upon the loop momentum and some sum of external momenta which does not vanish on-shell. For simplicity this will be referred to as a "massive" propagator, since this thesis considers only massless theories and the term is therefore unambiguous. For example,

$$\frac{1}{t_{l_1 56}} = \frac{1}{(l_1 + K_{56})^2} \,. \tag{3.2.1}$$

The presence of such a term has a major effect on the difficulty of evaluating the resulting canonical form, and as such it makes sense to divide canonical forms into two basic classes depending upon whether or not they contain such a propagator in the denominator. Canonical forms not containing a massive propagator are denoted by an H, or an $\mathcal{H}$ for the unintegrated form, and those with such a propagator factor are denoted by a G or $\mathcal{G}$.

In principle one could also encounter cut integrands containing multiple, non-equal massive propagators. One could simply treat these as a third class of canonical form and attempt to integrate accordingly, however adding another massive momentum to the problem greatly increases the difficulty of the integrand just as a G-function is a much more involved problem than an H-function. As such it is generally preferable to avoid any such integrals. This can if necessary be done by splitting the product of massive propagators into a sum of terms containing a single massive propagator using the spinor identities discussed in this chapter, however this proves unnecessary for the applications considered in this thesis: in gluon amplitudes with up to 7 external gluons, one can choose the form of the tree amplitudes used so as to avoid any such terms appearing.

The second type of loop momentum dependent structure which may appear in a canonical form is a pair of l-dependent spinor products in the numerator and denominator, e.g.

$$\frac{\langle a\, l\rangle}{\langle b\, l\rangle}\,. \tag{3.2.2}$$

Typically a canonical form will contain products of several such pairs, with the goal being that all are of the same type, with no mixture of angle and square spinor products or of l_1 and l_2 dependent pairs. We denote the number of such pairs present in the canonical form with a subscript number. One must then homogenize the typically mixed expressions of angle and square and l_1 and l_2 spinor products appearing in a typical cut integral to reduce it to the canonical form.

The final type of l-dependent structure which can appear in a canonical form is a spinor string with zero spinor weight in l but non-zero momentum weight, e.g.

$$[C|l|D\rangle\,. \tag{3.2.3}$$

We denote the number of such strings present in the numerator without a counterpart term in the denominator by a superscript. All required double-cut canonical forms can be constructed from these three structures. To illustrate the notation,

$$\begin{aligned}\mathcal{H}_1^2 =&[A|l|B\rangle[C|l|D\rangle\frac{\langle a\, l\rangle}{\langle b\, l\rangle}\,,\\ \mathcal{G}_2^1 =&\frac{[A|l|B\rangle}{t_{lQ}}[C|l|D\rangle\frac{\langle a_1\, l\rangle\,\langle a_2\, l\rangle}{\langle b_1\, l\rangle\,\langle b_2\, l\rangle}\,.\end{aligned} \tag{3.2.4}$$

In principle this notation is sufficient for all required canonical forms, however it is also useful to introduce an additional notation to denote canonical forms in which one or more spinor products in the denominator are identical, giving rise to a multiple pole. Since the presence of such a multiple pole has a significant impact on the evaluation of the integral and the resulting canonical form, the presence of a multiple pole is indicated by an x in the superscript, or multiple x's for triple or higher poles, e.g.

$$\begin{aligned}\mathcal{H}_1^{1;x} =&[A|l|B\rangle\frac{\langle a_1\,l\rangle\,\langle a_2\,l\rangle}{\langle b\,l\rangle^2}\,,\\ \mathcal{G}_1^{0;xx} =&[A|l|B\rangle\frac{\langle a_1\,l\rangle\,\langle a_2\,l\rangle\,\langle a_3\,l\rangle}{\langle b\,l\rangle^3}\,.\end{aligned}\tag{3.2.5}$$

With the minimal basis of structures we intend to construct all canonical forms out of thus defined, we must now consider how the much broader range of structures which can potentially appear in a cut can be reduced to this basis. This can be achieved through the application of a range spinor identities. The first of these, originally derived in this form by Britto and Feng [38] relies upon combining the Schouten identity with the technique of partial fractioning to allow a product of multiple pairs of l-dependent spinor products to be reduced to a sum of single pairs multiplied by non-l-dependent pieces,

$$\begin{aligned}\frac{\langle a_1\,l\rangle\,\langle a_2\,l\rangle}{\langle b_1\,l\rangle\,\langle b_2\,l\rangle} =&\frac{\langle a_1\,l\rangle\,\langle a_2\,l\rangle\,\langle b_1\,b_2\rangle}{\langle b_1\,l\rangle\,\langle b_2\,l\rangle\,\langle b_1\,b_2\rangle}\,,\\ =&\frac{-\,\langle a_1\,l\rangle\,(\langle a_2\,b_1\rangle\,\langle b_2\,l\rangle+\langle a_2\,b_2\rangle\,\langle l\,b_1\rangle)}{\langle b_1\,l\rangle\,\langle b_2\,l\rangle\,\langle b_1\,b_2\rangle}\,,\\ =&\,\langle a_1\,l\rangle\left(\frac{\langle a_2\,b_1\rangle}{\langle b_1\,l\rangle\,\langle b_2\,b_1\rangle}+\frac{\langle a_2\,b_2\rangle}{\langle b_2\,l\rangle\,\langle b_1\,b_2\rangle}\right)\,.\end{aligned}\tag{3.2.6}$$

This identity can be iterated systematically for longer products of pairs,

$$\prod_{i=1}^{n}\frac{\langle a_i\,l\rangle}{\langle b_i\,l\rangle} = \sum_{i}^{n}\frac{\langle a_1\,l\rangle\prod_{j=2}^{n}\langle a_j\,b_i\rangle}{\langle b_i\,l\rangle\prod_{j\neq i}^{n}\langle b_j\,b_i\rangle}\,.\tag{3.2.7}$$

This allows us to split up strings of spinor product pairs of the same type, and with the same dependency on l; however, most actual cuts will typically contain mixtures both of angle and square products and of functions of l_1 and l_2. We therefore require identities which will allow us to convert such mixed terms to expressions in terms of a single homogeneous type of spinor product, and dependent on a single loop momentum.

The simplest case to deal with is a mixture of square and angle spinor products. In this case, one can rewrite pairs of square angle products by multiplying top and bottom by an appropriate factor and applying momentum conservation across the cut,

$$
\begin{aligned}
\frac{[a\,l_1]}{[b\,l_1]} =& \frac{[a\,l_1]\,\langle l_1\,l_2\rangle}{[b\,l_1]\,\langle l_1\,l_2\rangle}\,,\\
=& \frac{[a|(l_1+l_2)|l_2\rangle}{[b|(l_1+l_2)|l_2\rangle}\,,\\
=& \frac{[a|P|l_2\rangle}{[b|P|l_2\rangle}\,,
\end{aligned}
\tag{3.2.8}
$$

where P is the cut momentum. Thus, we have converted a pair of l_1-dependent square products into a pair of l_2 dependent spinor strings, which can be treated for all intents and purposes as a pair of angle spinor products except for the fact that these spinor strings are symmetric under order reversal, unlike a simple spinor product which would be antisymmetric.

We can thus reduce an arbitrary mix of angle and square product pairs to a function of only angle product pairs, albeit in general with a mix of l_1 and l_2 dependent pairs. We therefore require an identity to homogenize such function in the loop momentum, producing a product of angle products dependent only upon one loop momentum. This is more difficult, although we can do so once again by introducing an appropriate factor in the numerator and denominator and applying momentum conservation,

$$
\begin{aligned}
\frac{\langle a\,l_2\rangle}{\langle b\,l_2\rangle} =& \frac{\langle a\,l_2\rangle\,[l_2|P|b\rangle}{\langle b\,l_2\rangle\,[l_2|P|b\rangle}\,,\\
=& \frac{\langle a|(P+l_1)P|b\rangle}{\langle b|(P+l_1)P|b\rangle}\,,\\
=& \frac{\langle a\,b\rangle\,P^2+\langle a\,l_1\rangle\,[l_1|P|b\rangle}{\langle b\,l_1\rangle\,[l_1|P|b\rangle}\,,\\
=& \frac{\langle a\,b\rangle\,P^2}{\langle b\,l_1\rangle\,[l_1|P|b\rangle}+\frac{\langle a\,l_1\rangle}{\langle b\,l_1\rangle}\,.
\end{aligned}
\tag{3.2.9}
$$

We can thus homogenize any expression in terms of l_1, at the cost of introducing terms which are subleading in overall power in l. It is important to note that these subleading terms do unfortunately include an effective square product in l_1; when converted to an angle product using equation (3.2.8) this will become an angle product in l_2 in the subleading term. We will thus have to apply equation (3.2.9) to homogenize it in l_1 again, introducing a sub-subleading term with the same problem. In principle the cycle could be iterated indefinitely, however fortunately we can terminate it thanks to our choice of Unitarity prescription. We choose to simply neglect any term appearing in the double cuts which when integrated will produce only a contribution to a triangle or box coefficient.

Specifically, this applies to any integrand with an overall power in l less than zero, since performing a double cut on a scalar triangle or box integral would give rise to an integrand still containing a loop propagator, multiplying a non-l dependent coefficient, thus with a momentum weight in l less than zero. In general we expect a double cut integrand term to be of overall power l^2, since it is a product of two tree amplitudes which must have total power of l^1 each. Thus for the case of the cuts of a complex scalar loop, we will expect to have to iterate equation (3.2.9) three times before we obtain a term which is of overall power of l^{-1}, which we can identify as being a contribution to a triangle coefficient and thus neglect. We thus obtain a mixture of terms with overall power in the loop momentum of l^2, l^1 and l^0.

In the case of an $\mathcal{N} = 1$ Supersymmetric multiplet loop, the situation is simpler. In this case the inherent numerator cancellations between the different particle types have the effect of reducing the total power in l by two, leaving no terms with total power greater than l^0 in the double cuts. This means any subleading term must automatically give rise only to triangle or box coefficients and can be discarded from the double cut, and as such homogenizing a string of spinor product pairs reduces to simply applying equation (3.2.8) and making the trivial replacement,

$$\frac{\langle a\, l_2\rangle}{\langle b\, l_2\rangle} \to \frac{\langle a\, l_1\rangle}{\langle b\, l_1\rangle}. \tag{3.2.10}$$

Aside from pairs of spinor products, other loop-momentum dependent structures appearing in the cut can generally be homogenized in l and reduced to one of the three canonical structures using simple momentum conservation and spinor identities. For example,

$$\begin{aligned} [A|l_2|B\rangle =& ([A|P|B\rangle + [A|l_1|B\rangle)\,,\\ \frac{1}{[A|(Q+l_2)|l_1\rangle} =& \frac{1}{[A|(P+Q+l_1)|l_1\rangle} = \frac{1}{[A|(P+Q)|l_1\rangle}\,,\\ \frac{1}{t_{Q(-l_2)}} =& \frac{1}{t_{(P-Q)l_1}}\,. \end{aligned} \tag{3.2.11}$$

3.3 Constructing the Canonical Basis

We can now postulate a minimum canonical basis for constructing double cut coefficients in massless Yang-Mills theory - we hope to construct all necessary double cuts using canonical forms dependent upon only one of the loop momenta, containing either one or zero massive loop-dependent propagators, and either one, two or no unpaired l-dependent spinor strings in the numerator, with an arbitrary number of numerator/denominator spinor product pairs.

The problem is thus one of how to solve this basis. The two simplest non-trivial canonical forms one can consider are those of a lone spinor product pair, and of a lone spinor string, denoted by our notation as $\mathcal{H}_1^0$ and $\mathcal{H}_0^1$ respectively. We will first consider $\mathcal{H}_1^0$,

$$\mathcal{H}_1^0(a, b, l_1) = \frac{\langle a\, l_1\rangle}{\langle b\, l_1\rangle}. \tag{3.3.1}$$

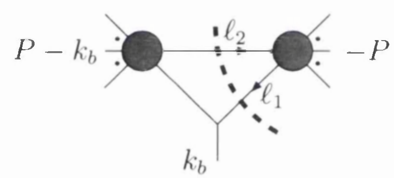
In accordance with the notation of the previous section, this corresponds to the integrand of the canonical form $H_1^0(a, b, P)$,

$$H_1^0(a, b, P)I_2(P) = \int dLIPS\mathcal{H}_1^0(a, b, l_1)\bigg|_{bubble}. \tag{3.3.2}$$

This can be evaluated by multiplying both top and bottom by a factor $[l_1\, b]$, such that the denominator can be interpreted as a propagator,

$$\begin{aligned} \mathcal{H}_1^0(a, b, l_1) =& \frac{[b|l_1|a\rangle}{[b|l_1|b\rangle}, \\ =& \frac{[b|l_1|a\rangle}{(l_1 - k_b)^2}. \end{aligned} \tag{3.3.3}$$

This has the form of a linear triangle with massless leg k_b,



Since this is the result obtained by performing a double cut on the linear triangle we can solve this canonical form by instead considering the unique triangle integral which yields this double cut, and solving for the coefficient of the scalar bubble function $I_2(P)$, where P is the cut momentum. We thus replace the cut integral with a standard covariant integral,

$$\int dLIPS \to \int \frac{d^d l_1}{l_1^2(l_1 + P)^2}, \tag{3.3.4}$$

thus the full cut integral becomes an ordinary loop integral,

$$\begin{aligned} \int dLIPS\frac{[b|l_1|a\rangle}{(l_1 - k_b)^2} \equiv& \int d^D l_1 \frac{[b|l_1|a\rangle}{l_1^2(l_1 + P)^2(l_1 - k_b)^2}\bigg|_{disc}, \\ =& [b|\gamma_\mu|a\rangle \int d^D l_1 \frac{l_1^\mu}{l_1^2(l_1 + P)^2(l_1 - k_b)^2}\bigg|_{disc}. \end{aligned} \tag{3.3.5}$$

The canonical form can thus be evaluated by decomposing the tensor integral on the second line via Passarino-Veltman reduction. The first step is to make a Lorentz projection in terms of the available tensors in the integral, namely P^μ and k_b^μ,

$$\int d^D l_1 \frac{l_1^\mu}{l_1^2(l_1+P)^2(l_1-k_b)^2} = P^\mu C_1 + k_b^\mu C_2 \,. \tag{3.3.6}$$

Ordinarily one would not be able to obtain an exact solution for the coefficients C_1 and C_2, but an important simplfication arises due to the fact that k_b is massless and on-shell. As such when we contract with k_b^μ, the C_2 term drops out since $k_b^2 = 0$ and we can solve for C_1,

$$\begin{aligned} C_1 =& \frac{1}{2P.k_b} \int d^D l_1 \frac{l_1.k_b}{l_1^2(l_1+P)^2(l_1-k_b)^2}\,, \\ =& \frac{1}{2P.k_b} \int d^D l_1 \frac{1}{l_1^2(l_1+P)^2}\,, \\ =& \frac{1}{2P.k_b} I_2(P)\,. \end{aligned} \tag{3.3.7}$$

We could in principle now contract equation (3.3.6) with P^μ in order to solve for C_2, however this is unnecessary in order to solve the canonical form, due to the dependence on $[a|\gamma^\mu|b\rangle$. When we contract equation (3.3.6) with this spinor string, we observe that we obtain

$$H_1^0(a,b,P)I_2(P) = C_1[a|P|b\rangle + C_2[a|k_b|b\rangle\,,$$

since k_b is on shell and massless the C_2 term vanishes due to equation (2.3.17), and we are left with

$$H_1^0(a,b,P)I_2(P) = \frac{[b|P|a\rangle}{2P.k_b} I_2(P)\,. \tag{3.3.8}$$

Since we have reduced this to a form consisting of an l-independent coefficient multiplying the scalar bubble integral, we can identify this coefficient as the bubble coefficient contribution of the $\mathcal{H}_1^0$ canonical form,

$$\begin{aligned} H_1^0(a,b,P) =& \frac{[b|P|a\rangle}{2P.k_b}\,, \\ =& \frac{[b|P|a\rangle}{[b|P|b\rangle}\,. \end{aligned} \tag{3.3.9}$$

We can ultimately extend this result to evaluate all of the $O(l^0)$ canonical basis without further integration. For example, one can note that we can apply equation (3.2.6) to reduce an arbitrarily long product of pairs of spinor products to a sum of H_1^0 functions; for example,

$$
\begin{aligned}
\mathcal{H}_2^0(A_1, A_2, B_1, B_2, l_1) &= \frac{\langle A_1\, l_1\rangle \langle A_2\, l_1\rangle}{\langle B_1\, l_1\rangle \langle B_2\, l_1\rangle}\,, \\
&= \frac{\langle A_2\, B_1\rangle \langle A_1\, l_1\rangle}{\langle B_2\, B_1\rangle \langle B_1\, l_1\rangle} + \frac{\langle A_2\, B_2\rangle \langle A_1\, l_1\rangle}{\langle B_1\, B_2\rangle \langle B_2\, l_1\rangle}\,, \\
H_2^0(A_1, A_2, B_1, B_2, P) &= \frac{\langle A_2\, B_1\rangle\, [A_1|P|B_1\rangle}{\langle B_2\, B_1\rangle\, [B_1|P|B_1\rangle} + \frac{\langle A_2\, B_2\rangle\, [A_1|P|B_2\rangle}{\langle B_1\, B_2\rangle\, [B_2|P|B_2\rangle}\,.
\end{aligned}
\tag{3.3.10}
$$

In addition to solving canonical forms by them splitting into H_1^0 functions, one can also evaluate them more directly. Rather than actually performing the integration, one can solve the coefficient by constructing an ansatz solution using the available parameters in the canonical form, reminiscent of the Passarino-Veltman approach to solving tensor loop integrals [1]. For relatively simple canonical forms this approach proves surprisingly effective; the small number of parameters in the problem, together with the constraints imposed by momentum conservation and the properties of spinors lead to a relatively small number of possible terms. For example, we can consider the simplest canonical form containing a double pole,

$$
\mathcal{H}_1^{0;x}(A_1, A_2, B, l_1) = \frac{\langle A_1\, l_1\rangle \langle A_2\, l_1\rangle}{\langle B\, l_1\rangle^2}\,. \tag{3.3.11}
$$

In this case we have only four parameters from which to construct candidate terms: the spinors A_1, A_2 and B, and the cut momentum P. We can also place several constraints upon possible terms which can be constructed.

1. The ansatz must have the same crossing symmetry as the canonical form it solves.
2. The ansatz must be of the same total momentum weight as the canonical form.
3. The ansatz must have the same overall spinor weight in each of the spinors as the canonical form.

These constraints severely limit the number of possible candidate terms which we can construct. For example, 1) eliminates from consideration any terms including the spinor product $\langle A_1\, A_2\rangle$; since the product is antisymmetric under interchange of A_1 and A_2, one necessarily cannot produce a symmetric term under the exchange of these spinors containing it. Similarly, 3) implies that we cannot simply introduce extra spinors into the problem, nor extra terms dependent on the permitted spinors if they would result in a term with different spinor weight to the canonical form. For instance, the term

$$\frac{P^2 \langle B\, A_2\rangle}{\langle B\, A_1\rangle}\,, \tag{3.3.12}$$

would be disallowed by 3), since it has the wrong spinor weight in both B and A_1, as well as being disallowed by 2) since it also has incorrect overall momentum weight thanks to the factor of P^2. One can, however, introduce additional factors of spinors providing they cancel in contribution to overall spinor weight in that spinor. For example, one can introduce extra factors of the spinor B provided one does so equivalently in the numerator and denominator, constructing a term

$$\frac{[B|P|A_1\rangle[B|P|A_2\rangle}{[B|P|B\rangle^2}\,, \tag{3.3.13}$$

which satisfies all three constraints. Since this is the only acceptable candidate term which can be constructed, it is a likely ansatz to be the entire solution to the canonical form; however, to confirm, we must check its behaviour under the limit that two of the spinors become collinear. In this case the integrand reduces to that of the $\mathcal{H}_1^0$ function,

$$\frac{\langle A_1\, l_1\rangle\,\langle A_2\, l_1\rangle}{\langle B\, l_1\rangle^2} \stackrel{A_1\to B}{\longrightarrow} \frac{\langle A_2\, l_1\rangle}{\langle B\, l_1\rangle} = \mathcal{H}_1^0(A_2, B, l_1)\,. \tag{3.3.14}$$

We can confirm that for the candidate expression, this limit reduces the expression to the H_1^0 canonical form,

$$\frac{[B|P|A_1\rangle[B|P|A_2\rangle}{[B|P|B\rangle^2} \stackrel{A_1\to B}{\longrightarrow} \frac{[B|P|A_2\rangle}{[B|P|B\rangle}\,. \tag{3.3.15}$$

Similarly, the limit $A_2 \to B$ produces an equivalent expression. Thus we have verified that our ansatz is the solution to the integral

$$H_1^{0;x}(A_1, A_2, B, P) = \frac{[B|P|A_1\rangle[B|P|A_2\rangle}{[B|P|B\rangle^2}\,. \tag{3.3.16}$$

This checking of the appropriate limits is necessary in order both to confirm that our candidate terms are of the correct form, and in order to fix any numerical coefficients and minus signs, which necessarily cannot be obtained from ansatz construction.

This technique can similarly be used to obtain the simplest canonical form containing a triple pole,

$$\mathcal{H}_1^{0;xx}(A_1, A_2, A_3, B, l_1) = \frac{\langle A_1\, l_1\rangle\,\langle A_2\, l_1\rangle\,\langle A_3\, l_1\rangle}{\langle B\, l_1\rangle^3}\,. \tag{3.3.17}$$

Again, the small number of available spinors results in only a single possible term for the canonical form,

$$H_1^{0;xx}(A_1, A_2, A_3, B, P) = \frac{[B|P|A_1\rangle[B|P|A_2\rangle[B|P|A_3\rangle}{[B|P|B\rangle^3}\,. \tag{3.3.18}$$

This expression reduces to an $H_1^{0;x}$ canonical form in the limit $A_i \to B$, thereby confirming it has the correct sign and numerical factor.

The canonical forms become more complicated as we start to consider cases which are linear in l, due to the larger number of spinors available with which to construct terms. The simplest possible $\mathcal{O}(l)$ canonical form is the $\mathcal{H}_0^1$,

$$\mathcal{H}_0^1(A, B, l_1) = [A|l_1|B\rangle\,. \tag{3.3.19}$$

This cannot be reduced to any of our so-far derived canonical forms by choice of an obvious limit, which makes it difficult to test any possible ansatz for it, but fortunately like the $\mathcal{H}_1^0$ it is relatively easy to evaluate directly due to its simplicity.

As with the $\mathcal{H}_1^0$, we identify the $\mathcal{H}_0^1$ canonical form as the discontinuity yielded by cutting a specific integral, in this case a linear bubble,

$$\begin{aligned}\int dLIPS[A|l_1|B\rangle &\equiv \int d^D l_1 \frac{[A|l_1|B\rangle}{l^2(l+P)^2}\bigg|_{disc}\,,\\ &= [A|\gamma_\mu|B\rangle \int d^D l_1 \frac{l_1^\mu}{l_1^2(l_1+P)^2}\bigg|_{disc}\,.\end{aligned} \tag{3.3.20}$$

One can show via a similar Passarino-Veltman reduction to that used to obtain $H_1^0(a, b, P)$ that the tensor integral is equivalent to $\frac{1}{2}P^\mu I_2^{[0]}(P)$, where $I_2^{[0]}(P)$ is the scalar bubble integral. The full integral is thus

$$\frac{[A|P|B\rangle}{2} I_2^{[0]}(P)\,. \tag{3.3.21}$$

We can thus identify the coefficient of the scalar bubble integral as the bubble contribution from the $\mathcal{H}_0^1$,

$$H_0^1(A, B, P, l_1) = \frac{1}{2}[A|P|B\rangle\,. \tag{3.3.22}$$

With the H_0^1 computed we can move on to deriving more complicated linear and quadratic canonical forms from it. The simplest such case is the canonical form $\mathcal{H}_1^1$,

$$\mathcal{H}_1^1(A, B, a, b, l_1) = [A|l_1|B\rangle \frac{\langle a\, l_1\rangle}{\langle b\, l_1\rangle}\,. \tag{3.3.23}$$

In this case, there are two possible terms, each consisting of two parts in order to have the correct symmetry under exchange of B and a; one with a factor of $[b|P|b\rangle$ in the denominator,

$$\frac{[A|P|B\rangle[b|P|a\rangle}{[b|P|b\rangle} + \frac{[A|P|a\rangle[b|P|B\rangle}{[b|P|b\rangle}, \tag{3.3.24}$$

and one with a factor of $[b|P|b\rangle^2$,

$$\frac{P^2[A|b|B\rangle[b|P|a\rangle}{[b|P|b\rangle^2} + \frac{P^2[A|b|a\rangle[b|P|B\rangle}{[b|P|b\rangle^2}. \tag{3.3.25}$$

In this case it is clear we need to consider the limits, both because the overall sign and numerical coefficient are unknown, and to obtain the relative sign between the two terms. We observe that under the limit $a \to b$, the integrand reduces to an $\mathcal{H}_0^1$ function,

$$\frac{[A|l_1|B\rangle\,\langle a\,l_1\rangle}{\langle b\,l_1\rangle} \xrightarrow{a\to b} \mathcal{H}_0^1(A,B,l_1)\,. \tag{3.3.26}$$

Taking this limit for our two terms, we obtain the expression

$$\begin{aligned}\frac{[A|P|B\rangle[b|P|a\rangle}{[b|P|b\rangle} + \frac{[A|P|a\rangle[b|P|B\rangle}{[b|P|b\rangle} &\to [A|P|B\rangle + \frac{[A|P|b\rangle[b|P|B\rangle}{[b|P|b\rangle} \\ \frac{P^2[A|b|B\rangle[b|P|a\rangle}{[b|P|b\rangle^2} + \frac{P^2[A|b|a\rangle[b|P|B\rangle}{[b|P|b\rangle^2} &\to \frac{P^2[A|b|B\rangle}{[b|P|b\rangle}\,.\end{aligned} \tag{3.3.27}$$

In order to reduce this to H_0^1 and thus fix the numerical coefficients, we first observe that the second term can be rewritten,

$$\frac{P^2[A|b|B\rangle}{[b|P|b\rangle} = \frac{[A|PP|b]\,\langle b\,B\rangle}{[b|P|b\rangle}. \tag{3.3.28}$$

In this form, we can use the Schouten identity to contract this term with the second part of the first term,

$$\begin{aligned}\frac{[A|PP|b]\,\langle b\,B\rangle - [A|P|b\rangle[b|P|B\rangle}{[b|P|b\rangle} &= -\,\frac{[A|P|B\rangle[b|P|b\rangle}{[b|P|b\rangle}, \\ &= -\,[A|P|B\rangle\,.\end{aligned} \tag{3.3.29}$$

The total limit with this sign between the terms is thus equal to $-2[A|P|B\rangle$. It is therefore clear that in order to obtain the correct limit of H_0^1, we require an overall sign of $-\frac{1}{4}$, resulting in the full canonical form being,

$$\begin{aligned}H_1^1(A,B,a,b,P,l_1) = &-\,\frac{[A|P|B\rangle[b|P|a\rangle + [A|P|a\rangle[b|P|B\rangle}{4[b|P|b\rangle} \\ &-\,\frac{P^2[A|b|B\rangle[b|P|a\rangle + P^2[A|b|a\rangle[b|P|B\rangle}{4[b|P|b\rangle^2}\,.\end{aligned} \tag{3.3.30}$$

We also need to solve the case of an H-function which is quadratic in l_1, the $\mathcal{H}_1^2$ function,

$$\mathcal{H}_1^2(A,B,C,D,a,b,l_1) = [A|l_1|B\rangle[C|l_1|D\rangle\frac{\langle a\, l_1\rangle}{\langle b\, l_1\rangle}\,. \tag{3.3.31}$$

This, again, can be solved by the ansatz approach. As with the $\mathcal{H}_1^1$ case, the presence of only one spinor in the denominator limits the number of possible terms, in this case to three, since we can add up to two factors of $[b|P|b\rangle$ in the denominator before we run out of suitable terms to pair the resulting spinors with in the numerator. The three distinct terms each have up to twelve symmetry permutations, due to the multiple ways of pairing up the two square spinors A and C with the angles B, D and a. The terms are given by

$$\begin{aligned}
&1)\frac{[A|P|B\rangle[C|P|D\rangle[b|P|a\rangle}{[b|P|b\rangle} + 5 \text{ other permutations}\,,\\
&2)\frac{P^2[A|P|B\rangle[C|b|D\rangle[b|P|a\rangle}{[b|P|b\rangle^2} + 11 \text{ other permutations}\,,\\
&3)\frac{P^4[A|b|B\rangle[C|b|D\rangle[b|P|a\rangle}{[b|P|b\rangle^3} + 6 \text{ other permutations}\,.
\end{aligned} \tag{3.3.32}$$

Again it is necessary to examine the limits of this canonical form in order to extract the numerical coefficients and signs. In this case the limit $a \to b$ should reproduce the canonical form H_1^1. This process yields the coefficients,

$$\begin{aligned}
H_1^2(A,B,C,D,a,b,P) =&\frac{[A|P|B\rangle[C|P|D\rangle[b|P|a\rangle}{18[b|P|b\rangle} + \text{Permutations}(B,D,a)\\
&+\frac{P^2[A|P|B\rangle[C|b|D\rangle[b|P|a\rangle}{36[b|P|b\rangle^2} + \text{Permutations}(B,D,a;A,C)\\
&+\frac{P^4[A|b|B\rangle[C|b|D\rangle[b|P|a\rangle}{18[b|P|b\rangle^3} + \text{Permutations}(B,D,a)\,.
\end{aligned} \tag{3.3.33}$$

We will also need to solve the case of a linear H-functions with a double pole, the $\mathcal{H}_1^{1;x}$,

$$\mathcal{H}_1^{1;x}(A,B,a_1,a_2,b,l_1) = [A|l_1|B\rangle\frac{\langle a_1\, l_1\rangle\,\langle a_2\, l_1\rangle}{\langle b\, l_1\rangle^2}\,. \tag{3.3.34}$$

The ansatz method yields the expression for this,

$$\begin{aligned}
H_1^{1;x}(A,B,a_1,a_2,b,P) =&\frac{[A|P|B\rangle[b|P|a_1\rangle[b|P|a_2\rangle}{6[b|P|b\rangle^2} + \text{Permutations}(B,a_1,a_2)\\
&+\frac{P^2[A|b|B\rangle[b|P|a_1\rangle[b|P|a_2\rangle}{3[b|P|b\rangle^3} + \text{Permutations}(B,a_1,a_2)\,.
\end{aligned} \tag{3.3.35}$$

The final H-function canonical form required is that containing two separate double poles, which we denote as an $\mathcal{H}_1^{0;xy}$ function,

$$\mathcal{H}_1^{0;xy}(a_1,a_2,a_3,a_4,b_1,b_2) = \frac{\langle a_1\, l_1\rangle \langle a_2\, l_1\rangle \langle a_3\, l_1\rangle \langle a_4\, l_1\rangle}{\langle b_1\, l_1\rangle^2 \langle b_2\, l_1\rangle^2}\,. \tag{3.3.36}$$

In principle this could be evaluated using the ansatz method, however this is unnecessary since we can instead simply split the poles in order to obtain a sum of simpler H-functions, which has the benefit that the analysis could be generalized to any order in l.

$$\begin{aligned}
\mathcal{H}_1^{0;xy} &= \langle a_1\, l_1\rangle \langle a_2\, l_1\rangle \left(\frac{\langle a_3\, b_1\rangle}{\langle b_1\, l_1\rangle \langle b_2\, l_1\rangle} + \frac{\langle a_3\, b_2\rangle}{\langle b_1\, b_2\rangle \langle b_2\, l_1\rangle}\right)\left(\frac{\langle a_4\, b_1\rangle}{\langle b_1\, l_1\rangle \langle b_2\, b_1\rangle} + \frac{\langle a_4\, b_2\rangle}{\langle b_1\, b_2\rangle \langle b_2\, l_1\rangle}\right),\\
&= -\langle a_1\, l_1\rangle \left(\frac{\langle a_2\, b_1\rangle}{\langle b_1\, l_1\rangle \langle b_2\, b_1\rangle} + \frac{\langle a_2\, b_2\rangle}{\langle b_1\, b_2\rangle \langle b_2\, l_1\rangle}\right)\frac{\langle a_3\, b_1\rangle \langle a_4\, b_2\rangle + \langle a_3\, b_2\rangle \langle a_4\, b_1\rangle}{\langle b_2\, b_1\rangle^2}\\
&\quad + \frac{\langle a_1\, l_1\rangle \langle a_2\, l_1\rangle \langle a_3\, b_1\rangle \langle a_4\, b_1\rangle}{\langle b_1\, l_1\rangle^2 \langle b_2\, b_1\rangle^2} + \frac{\langle a_1\, l_1\rangle \langle a_2\, l_1\rangle \langle a_3\, b_2\rangle \langle a_4\, b_2\rangle}{\langle b_2\, l_1\rangle^2 \langle b_2\, b_1\rangle^2}\\
H_1^{0;xy}&(a_1,a_2,a_3,a_4,b_1,b_2,P) =\\
&\frac{\langle a_3\, b_1\rangle \langle a_4\, b_2\rangle + \langle a_3\, b_2\rangle \langle a_4\, b_1\rangle}{\langle b_2\, b_1\rangle^3}\left(\langle a_2\, b_1\rangle H_1^0(a_1,b_1,P) - \langle a_2\, b_2\rangle H_1^0(a_1,b_2,P)\right)\\
&+\frac{\langle a_3\, b_1\rangle \langle a_4\, b_1\rangle}{\langle b_2\, b_1\rangle^2} H_1^{0;x}(a_1,a_2,b_1,P) + \frac{\langle a_3\, b_2\rangle \langle a_4\, b_2\rangle}{\langle b_2\, b_1\rangle^2} H_1^{0;x}(a_1,a_2,b_2,P)\,.
\end{aligned} \tag{3.3.37}$$

3.4 Canonical Forms with "Massive" Propagators

We must now consider how to evaluate the harder class of canonical forms containing a massive l-dependent propagator, the G functions. The simplest such case which can contribute to a bubble coefficient is the function $\mathcal{G}_0^0$,

$$\mathcal{G}_0^0(A;B;Q;l_1) = \frac{[A|l_1|B\rangle}{(l_1+Q)^2}\,, \tag{3.4.1}$$

where $Q^2 \neq 0$. In principle, we could try to tackle this canonical form using the ansatz approach; however, it soon becomes apparent that one obtains a surprisingly large number of candidate terms by doing so. This is primarily due to the presence of a second non-null momentum in the problem, Q. The presence of this momentum greatly increases the number of possible kinematic structures which can be constructed; with only one non-null momentum, we can essentially only construct three structures.

- Simple spinor products: $\langle A\, B\rangle$, $[A\, B]$.
- Spinor strings of order 2: $[A|P|B\rangle$.

- The cut momentum invariant: P^2.

However, with the momentum Q to work with, many more structures become possible: $P.Q$, Q^2, $\langle A|PQ|B\rangle$, $[A|QPQ|B\rangle$ etc. This suggests that the ansatz approach is not nearly as well suited to evaluating G-functions as it is to H-functions, especially when we consider higher-order G-functions with large numbers of spinors.

Therefore instead of trying to compute these nontrivial integrals directly or by positing an ansatz, we can attempt to rewrite them in terms of our previously solved H-function canonical basis. For the $\mathcal{G}_0^0$ we begin by exploiting the identity

$$\frac{[A|l|B\rangle}{(l+Q)^2} = \frac{\langle l\,B\rangle}{(l+Q)^2}\frac{[A|P(P+Q)Q|l\rangle}{\langle l|PQ|l\rangle} - \frac{\langle l\,B\rangle\,[A|P|l\rangle}{\langle l|PQ|l\rangle}\,. \tag{3.4.2}$$

We can now observe that the first term in this expression is one overall power in l lower than the original integrand, and since the original $\mathcal{G}_0^0$ function is of order l^0, this term can be discarded from the calculation since it does not contribute to the bubble coefficients. We are thus left with

$$\mathcal{G}_0^0|_{bubble} = -\frac{\langle l_1\,B\rangle\,[A|P|l_1\rangle}{\langle l_1|PQ|l_1\rangle}\,. \tag{3.4.3}$$

This simpler denominator can be now be rewritten in a form which can be split apart, providing we can find some momenta $\hat{P}$ and $\hat{Q}$ which are both null and have the property

$$\langle l_1|\hat{P}\hat{Q}|l_1\rangle = K\langle l_1|PQ|l_1\rangle\,. \tag{3.4.4}$$

Generally such momenta can be found by taking a linear combination of P and Q and finding the appropriate coefficients such that the combination becomes null. In particular the following choice of $\hat{P}$ and $\hat{Q}$ originally found by Forde [19], satisfies the condition and is the one we choose to apply,

$$\begin{aligned}\hat{P}^\mu =& \frac{1}{2\sqrt{\Delta_3}}\left(P^2Q^\mu - \left(P.Q - \frac{\sqrt{\Delta_3}}{2}\right)P^\mu\right)\,,\\ \hat{Q}^\mu =& \frac{1}{2\sqrt{\Delta_3}}\left(-P^2Q^\mu + \left(P.Q + \frac{\sqrt{\Delta_3}}{2}\right)P^\mu\right)\,,\end{aligned} \tag{3.4.5}$$

where the factor $\Delta_3 = 4(P.Q)^2 - 4P^2Q^2$ is the Gram determinant of the three mass triangle integral with legs of momentum P, Q and $-P-Q$. With this choice of $\hat{P}$ and $\hat{Q}$, we now have the relation

$$\langle l_1|PQ|l_1\rangle = -\frac{4\sqrt{\Delta_3}}{P^2}\langle l_1|\hat{P}\hat{Q}|l_1\rangle\,. \tag{3.4.6}$$

More generally we can derive various useful identities for relating $\hat{P}$ and $\hat{Q}$ dependent objects to the other kinematic variables.

Object	Expanded form
$\hat{P}^2$	0
$\hat{Q}^2$	0
$P.\hat{Q}$	$\frac{P^2}{4}$
$P.\hat{P}$	$\frac{P^2}{4}$
$Q.\hat{Q}$	$\frac{\sqrt{\Delta_3}}{8}+\frac{P.Q}{4}$
$Q.\hat{P}$	$-\frac{\sqrt{\Delta_3}}{8}+\frac{P.Q}{4}$
$\hat{P}.\hat{Q}$	$\frac{P^2}{8}$
$\langle A\vert\hat{P}\hat{Q}\vert A\rangle$	$-\frac{P^2}{4\sqrt{\Delta_3}}\langle A\vert PQ\vert A\rangle$
$\langle A\vert\hat{P}\hat{Q}\vert B\rangle$	$\frac{1}{4\Delta_3}\left(-P^2\sqrt{\Delta_3}\langle A\vert PQ\vert B\rangle+\left(\frac{P^2\Delta_3}{2}+\sqrt{\Delta_3}P^2(P.Q)\right)\langle A\,B\rangle\right)$
$\frac{\langle A\,\hat{Q}\rangle}{\langle B\,\hat{Q}\rangle}$	$\frac{\langle A\vert QP\vert B\rangle-(P.Q+\frac{\sqrt{\Delta_3}}{2})\langle A\,B\rangle}{\langle B\vert QP\vert B\rangle}$
$\frac{\langle A\,\hat{P}\rangle}{\langle B\,\hat{P}\rangle}$	$\frac{\langle A\vert QP\vert B\rangle-(P.Q-\frac{\sqrt{\Delta_3}}{2})\langle A\,B\rangle}{\langle B\vert QP\vert B\rangle}$

Figure 3.1: Conversion table for $\hat{P}$ and $\hat{Q}$ dependent quantities

Since $\hat{P}$ and $\hat{Q}$ are null vectors, we can thus express the $\mathcal{G}_0^0$ function in the form,

$$\mathcal{G}_0^0|_{bubble}=\frac{P^2\,\langle l_1\,B\rangle\,[A|P|l_1\rangle}{4\sqrt{\Delta_3}\langle l_1|\hat{P}\hat{Q}|l_1\rangle}=\frac{P^2\,\langle l_1\,B\rangle\,[A|P|l_1\rangle}{4\sqrt{\Delta_3}\langle l_1\,\hat{P}\rangle[\hat{P}\,\hat{Q}]\langle\hat{Q}\,l_1\rangle}\,. \tag{3.4.7}$$

This expression can now be split using equation (3.2.6),

$$\begin{aligned}G_0^0|_{bubble}=&\frac{P^2\langle B\,\hat{P}\rangle[A|P|l_1\rangle}{4\sqrt{\Delta_3}[\hat{P}\,\hat{Q}]\langle\hat{Q}\,\hat{P}\rangle\langle\hat{P}\,l_1\rangle}-\frac{P^2\langle B\,\hat{Q}\rangle[A|P|l_1\rangle}{4\sqrt{\Delta_3}[\hat{P}\,\hat{Q}]\langle\hat{Q}\,\hat{P}\rangle\langle\hat{Q}\,l_1\rangle},\\=&\frac{\langle B\,\hat{P}\rangle}{\sqrt{\Delta_3}}\mathcal{H}_1^0([A|P,\hat{P},l_1)-\frac{\langle B\,\hat{Q}\rangle}{\sqrt{\Delta_3}}\mathcal{H}_1^0([A|P,\hat{Q},l_1)\,.\end{aligned} \tag{3.4.8}$$

We can therefore insert the known expressions for the H_1^0 canonical forms,

$$\begin{aligned}G_0^0=&-\frac{\langle B\,\hat{P}\rangle[\hat{P}\,A]P^2}{\sqrt{\Delta_3}[\hat{P}|P|\hat{P}\rangle}+\frac{P^2\langle B\,\hat{Q}\rangle[\hat{Q}\,A]}{\sqrt{\Delta_3}[\hat{Q}|P|\hat{Q}\rangle},\\=&-\left(\frac{[A|Q|B\rangle P^2-[A|P|B\rangle\left(P.Q-\frac{\sqrt{\Delta_3}}{2}\right)}{\Delta_3}\right)+\left(\frac{-[A|Q|B\rangle P^2+[A|P|B\rangle\left(P.Q+\frac{\sqrt{\Delta_3}}{2}\right)}{\Delta_3}\right),\\=&\frac{-2P^2[A|Q|B\rangle+2P.Q[A|P|B\rangle}{\Delta_3}\,.\end{aligned} \tag{3.4.9}$$

We can apply the Schouten identity to the second term to obtain

$$G_0^0 = \frac{-2[A|PPQ|B\rangle + [A|PPQ|B\rangle + [A|PQP|B\rangle}{\Delta_3}, \\ = \frac{[A|P[Q,P]|B\rangle}{\Delta_3}. \tag{3.4.10}$$

Note that the square roots of the Gram determinant Δ_3 which we had introduced in the definition of $\hat{P}$ and $\hat{Q}$ have cancelled out to leave a final expression which is fully rational.

We next consider the case of the $\mathcal{G}_1^0$ canonical form,

$$\mathcal{G}_1^0(A,B,C,D,Q,l_1) = \frac{[A|l_1|B\rangle\langle l_1\,C\rangle}{(l_1+Q)^2\langle l_1\,D\rangle}. \tag{3.4.11}$$

We can introduce the $\hat{P}$ and $\hat{Q}$ dependent denominators as before, since this canonical form is also of total power zero in l and equation (3.4.2) thus returns only one term contributing to the bubble coefficients,

$$\mathcal{G}_1^0|_{bubble} = \frac{P^2\langle l_1\,B\rangle[A|P|l_1\rangle\langle l_1\,C\rangle}{4\sqrt{\Delta_3}\langle l_1\,\hat{P}\rangle[\hat{P}\,\hat{Q}]\langle\hat{Q}\,l_1\rangle\langle l_1\,D\rangle}. \tag{3.4.12}$$

Since there are now three l-dependent spinor products in the denominator we must split it twice, while keeping the explicit symmetry under interchange of C and B.

$$\begin{aligned}\mathcal{G}_1^0|_{bubble} =& -\frac{[A|P|l_1\rangle\langle\hat{P}\,C\rangle}{2\sqrt{\Delta_3}}\left(\frac{\langle D\,B\rangle}{\langle l_1\,D\rangle\langle D\,\hat{P}\rangle} + \frac{\langle\hat{P}\,B\rangle}{\langle\hat{P}\,D\rangle\langle l_1\,\hat{P}\rangle}\right) \\ &+ \frac{[A|P|l_1\rangle\langle\hat{Q}\,C\rangle}{2\sqrt{\Delta_3}}\left(\frac{\langle D\,B\rangle}{\langle l_1\,D\rangle\langle D\,\hat{Q}\rangle} + \frac{\langle\hat{Q}\,B\rangle}{\langle\hat{Q}\,D\rangle\langle l_1\,\hat{Q}\rangle}\right) + \{B\longleftrightarrow C\}, \\ =& -\left(\frac{\langle\hat{P}\,C\rangle\langle D\,B\rangle}{2\sqrt{\Delta_3}\langle D\,\hat{P}\rangle}\mathcal{H}_1^0([A|P,D,l_1) + \frac{\langle\hat{P}\,C\rangle\langle\hat{P}\,B\rangle}{2\sqrt{\Delta_3}\langle\hat{P}\,D\rangle}\mathcal{H}_1^0([A|P,\hat{P},l_1)\right) \\ &+ \left(\frac{\langle\hat{Q}\,C\rangle\langle D\,B\rangle}{2\sqrt{\Delta_3}\langle D\,\hat{Q}\rangle}\mathcal{H}_1^0([A|P,D,l_1) + \frac{\langle\hat{Q}\,C\rangle\langle\hat{Q}\,B\rangle}{2\sqrt{\Delta_3}\langle\hat{Q}\,D\rangle}\mathcal{H}_1^0([A|P,\hat{Q},l_1)\right) + \{B\longleftrightarrow C\}.\end{aligned} \tag{3.4.13}$$

Inserting the expressions for the H_1^0 canonical form, we can thus obtain an expression which splits naturally into two parts,

$$\begin{aligned}G_1^0 =& \left(\frac{\langle\hat{P}\,C\rangle}{\langle\hat{P}\,D\rangle} - \frac{\langle\hat{Q}\,C\rangle}{\langle\hat{Q}\,D\rangle}\right)\frac{\langle D\,B\rangle[D\,A]P^2}{2\sqrt{\Delta_3}[D|P|D\rangle} + \{B\longleftrightarrow C\} \quad (G_{1\alpha}^0) \\ &+ \frac{\langle\hat{P}\,C\rangle[A|\hat{P}|B\rangle}{\langle\hat{P}\,D\rangle\sqrt{\Delta_3}} - \frac{\langle\hat{Q}\,C\rangle[A|\hat{Q}|B\rangle}{\langle\hat{Q}\,D\rangle\sqrt{\Delta_3}} + \{B\longleftrightarrow C\} \quad (G_{1\beta}^0).\end{aligned} \tag{3.4.14}$$

One can make this division into terms with similar dependence upon $\hat{P}$ and $\hat{Q}$ in order to more easily identify cancellations of irrational terms and obtain a canonical form explicitly free of unphysical square roots of Gram determinants. $G^0_{1,\alpha}$ is the simpler term,

$$\begin{aligned}
G^0_{1,\alpha} =& \left(\frac{\langle C|QP|D\rangle - (P.Q - \frac{\sqrt{\Delta_3}}{2})\langle C\,D\rangle - \langle C|QP|D\rangle + (P.Q + \frac{\sqrt{\Delta_3}}{2})\langle C\,D\rangle}{\langle D|QP|D\rangle}\right) \frac{\langle B\,D\rangle[D\,A]P^2}{2\sqrt{\Delta_3}[D|P|D\rangle} \\
&+ \{B \longleftrightarrow C\}, \\
=& \frac{\langle C\,D\rangle\langle B\,D\rangle[D\,A]P^2}{[D|P|D\rangle\langle D|QP|D\rangle}.
\end{aligned} \tag{3.4.15}$$

Note that we obtain identical expressions from both of the symmetric terms. The $G^0_{1,\beta}$ term gives

$$\begin{aligned}
G^0_{1,\beta} =& \frac{1}{2\Delta_3}(P^2[A|Q|B\rangle - (P.Q - \frac{\sqrt{\Delta_3}}{2}[A|P|B\rangle)\frac{\langle C|QP|D\rangle - (P.Q - \frac{\sqrt{\Delta_3}}{2})\langle C\,D\rangle}{\langle D|QP|D\rangle} \\
&- \frac{1}{2\Delta_3}(-P^2[A|Q|B\rangle + (P.Q + \frac{\sqrt{\Delta_3}}{2}[A|P|B\rangle)\frac{\langle C|QP|D\rangle - (P.Q + \frac{\sqrt{\Delta_3}}{2})\langle C\,D\rangle}{\langle D|QP|D\rangle} + \{B \longleftrightarrow C\}, \\
=& \frac{P^2[A|Q|B\rangle\langle C|QP|D\rangle - P^2 P.Q[A|Q|B\rangle\langle C\,D\rangle}{\Delta_3\langle D|QP|D\rangle} \\
&+ \frac{-P.Q[A|P|B\rangle\langle C|QP|D\rangle + ((P.Q)^2 + \frac{\Delta_3}{4})[A|P|B\rangle\langle C\,D\rangle}{\Delta_3\langle D|QP|D\rangle} \\
&+ \{B \longleftrightarrow C\}.
\end{aligned} \tag{3.4.16}$$

Again, one can simplify this by absorbing the factors of $P.Q$ into the spinor strings using the Schouten identity,

$$G^0_{1,\beta} = \frac{[A|P[P,Q]|B\rangle\langle C|[P,Q]|D\rangle}{4\Delta_3\langle D|QP|D\rangle} + \frac{[A|P|B\rangle\langle C\,D\rangle[D|P|D\rangle}{4\langle D|QP|D\rangle[D|P|D\rangle} + \{B \longleftrightarrow C\}. \tag{3.4.17}$$

The full term is given by

$$\begin{aligned}
G^1_0 =& \frac{[A|P[Q,P]|B\rangle\langle C|[P,Q]|D\rangle}{4\Delta_3\langle D|PQ|D\rangle} + \frac{[A|P[Q,P]|C\rangle\langle B|[P,Q]|D\rangle}{4\Delta_3\langle D|PQ|D\rangle} \\
&+ \frac{[A|P|B\rangle\langle C\,D\rangle[D|P|D\rangle}{4\langle D|QP|D\rangle[D|P|D\rangle} + \frac{[A|P|C\rangle\langle B\,D\rangle[D|P|D\rangle}{4\langle D|QP|D\rangle[D|P|D\rangle} \\
&+ \frac{\langle C\,D\rangle\langle B\,D\rangle[A|PP|D]}{2\langle D|QP|D\rangle[D|P|D\rangle}.
\end{aligned} \tag{3.4.18}$$

Once again one can simplify this further using the Schouten identity,

$$G_1^0(A,B,C,D,P) = -\frac{[A|P[P,Q]|B\rangle\langle C|[P,Q]|D\rangle}{2\Delta_3\langle D|PQ|D\rangle} + \frac{[A|P|D\rangle([D|P|C\rangle\langle B\,D\rangle + [D|P|B\rangle\langle C\,D\rangle)}{2\langle D|PQ|D\rangle[D|P|D\rangle}. \tag{3.4.19}$$

We can apply the same technique to the problem of the order-l^0 $\mathcal{G}$-functions with multiple poles. Considering first the double-pole case, the $\mathcal{G}_1^{0;x}$ function,

$$\mathcal{G}_1^{0;x}(A,B,C,D,E,Q,l_1) = \frac{[A|l_1|B\rangle\langle l_1\,C\rangle\langle l_1\,D\rangle}{(l_1+Q)^2\langle l_1\,E\rangle^2}. \tag{3.4.20}$$

Applying the identity (3.4.2) as before, we obtain the contribution to the bubble coefficient

$$\mathcal{G}_1^{0;x}|_{bubble} = \frac{[A|P|l_1\rangle\langle l_1\,B\rangle\langle l_1\,C\rangle\langle l_1\,D\rangle}{\langle l_1|PQ|l_1\rangle\langle l_1\,E\rangle^2}. \tag{3.4.21}$$

Substituting in $\hat{P}$ and $\hat{Q}$ as usual,

$$\mathcal{G}_1^{0;x}|_{bubble} = \frac{P^2[A|P|l_1\rangle\langle l_1\,B\rangle\langle l_1\,C\rangle\langle l_1\,D\rangle}{4\sqrt{\Delta_3}[\hat{P}\,\hat{Q}]\langle l_1\,\hat{P}\rangle\langle l_1\,\hat{Q}\rangle\langle l_1\,E\rangle^2}, \tag{3.4.22}$$

we now proceed to split poles, attempting to keep the expression as symmetric as possible between $\hat{P}$ and $\hat{Q}$ in order to make it simpler to cancel the irrational terms once we have solved the canonical form.

$$\begin{aligned}
\mathcal{G}_1^{0;x}|_{bubble} =& \frac{P^2[A|P|l_1\rangle\langle l_1\,B\rangle\langle l_1\,C\rangle}{4\sqrt{\Delta_3}[\hat{P}\,\hat{Q}]\langle l_1\,E\rangle^2}\left(\frac{\langle\hat{P}\,D\rangle}{\langle l_1\,\hat{P}\rangle\langle\hat{P}\,\hat{Q}\rangle} + \frac{\langle\hat{Q}\,D\rangle}{\langle l_1\,\hat{Q}\rangle\langle\hat{Q}\,\hat{P}\rangle}\right), \\
=& \frac{P^2[A|P|l_1\rangle\langle l_1\,B\rangle\langle l_1\,C\rangle}{4\sqrt{\Delta_3}[\hat{P}\,\hat{Q}]\langle l_1\,E\rangle}\left(\frac{\langle\hat{P}\,D\rangle\langle\hat{P}\,C\rangle}{\langle l_1\,\hat{P}\rangle\langle\hat{P}\,\hat{Q}\rangle\langle\hat{P}\,E\rangle} + \frac{\langle\hat{P}\,D\rangle\langle E\,C\rangle}{\langle E\,\hat{P}\rangle\langle\hat{P}\,\hat{Q}\rangle\langle l_1\,E\rangle}\right. \\
&\left.\frac{+\langle\hat{Q}\,D\rangle\langle\hat{Q}\,C\rangle}{\langle l_1\,\hat{Q}\rangle\langle\hat{Q}\,\hat{P}\rangle\langle\hat{Q}\,E\rangle} + \frac{\langle\hat{Q}\,D\rangle\langle E\,C\rangle}{\langle E\,\hat{Q}\rangle\langle\hat{Q}\,\hat{P}\rangle\langle l_1\,E\rangle}\right), \\
=& \frac{P^2}{4\sqrt{\Delta_3}[\hat{P}\,\hat{Q}]}\left(\frac{[A|P|l_1\rangle\langle l_1\,B\rangle}{\langle l_1\,E\rangle^2}\left(\frac{\langle\hat{P}\,D\rangle\langle E\,C\rangle}{\langle E\,\hat{P}\rangle\langle\hat{P}\,\hat{Q}\rangle} + \frac{\langle\hat{Q}\,D\rangle\langle E\,C\rangle}{\langle E\,\hat{Q}\rangle\langle\hat{Q}\,\hat{P}\rangle}\right)\right. \\
&+[A|P|l_1\rangle\left(\frac{\langle\hat{P}\,D\rangle\langle\hat{P}\,C\rangle\langle\hat{P}\,B\rangle}{\langle l_1\,\hat{P}\rangle\langle\hat{P}\,\hat{Q}\rangle\langle\hat{P}\,E\rangle^2} + \frac{\langle\hat{Q}\,D\rangle\langle\hat{Q}\,C\rangle\langle\hat{Q}\,B\rangle}{\langle l_1\,\hat{Q}\rangle\langle\hat{Q}\,\hat{P}\rangle\langle\hat{Q}\,E\rangle^2}\right. \\
&\left.\left.-\frac{\langle\hat{P}\,D\rangle\langle\hat{P}\,C\rangle\langle E\,B\rangle}{\langle E\,\hat{P}\rangle^2\langle\hat{P}\,\hat{Q}\rangle\langle l_1\,E\rangle} - \frac{\langle\hat{Q}\,D\rangle\langle\hat{Q}\,C\rangle\langle E\,B\rangle}{\langle E\,\hat{Q}\rangle^2\langle\hat{Q}\,\hat{P}\rangle\langle l_1\,E\rangle}\right)\right).
\end{aligned} \tag{3.4.23}$$

This splits naturally into three pieces, for each of which we expect the irrational terms to cancel internally, since they each consist of a sum of a term dependent only upon $\hat{P}$ and one dependent only upon $\hat{Q}$. Starting with the simplest piece,

$$\mathcal{G}_{1,\alpha}^{0;x} = \frac{[A|P|l_1\rangle\langle l_1\,B\rangle\langle E\,C\rangle}{\sqrt{\Delta_3}\langle l_1\,E\rangle^2}\left(\frac{\langle\hat{P}\,D\rangle}{\langle\hat{P}\,E\rangle} - \frac{\langle\hat{Q}\,D\rangle}{\langle\hat{Q}\,E\rangle}\right). \tag{3.4.24}$$

We can solve this by inserting our solution for the $H_1^{0;x}$ canonical form, equation (3.3.16).

$$G_{1,\alpha}^{0;x} = \frac{1}{\sqrt{\Delta_3}}\frac{P^2[A\,E][E|P|B\rangle\langle E\,C\rangle}{[E|P|E\rangle^2}\left(\frac{\langle\hat{P}\,D\rangle}{\langle\hat{P}\,E\rangle} - \frac{\langle\hat{Q}\,D\rangle}{\langle\hat{Q}\,E\rangle}\right). \tag{3.4.25}$$

It is now trivial to expand out the $\hat{P}$ and $\hat{Q}$-dependent terms in order to cancel the irrational pieces,

$$\begin{aligned} G_{1,\alpha}^{0;x} =& \frac{1}{\sqrt{\Delta_3}}\frac{P^2[A\,E][E|P|B\rangle\langle E\,C\rangle}{[E|P|E\rangle^2}\frac{\sqrt{\Delta_3}\langle D\,E\rangle}{\langle E|QP|E\rangle}, \\ =& \frac{P^2[E\,A][E|P|B\rangle\langle D\,E\rangle\langle C\,E\rangle}{[E|P|E\rangle^2\langle E|QP|E\rangle}. \end{aligned} \tag{3.4.26}$$

The β term is a sum of $\mathcal{H}_1^0$ functions,

$$\begin{aligned} G_{1,\beta}^{0;x} =& \frac{1}{\sqrt{\Delta_3}}\frac{P^2[A\,E]\langle E\,B\rangle(\langle D|QP|E\rangle - (P.Q - \frac{\sqrt{\Delta_3}}{2})\langle D\,E\rangle)(\langle C|QP|E\rangle - (P.Q - \frac{\sqrt{\Delta_3}}{2})\langle C\,E\rangle)}{[E|P|E\rangle\langle E|QP|E\rangle^2} \\ &+ \frac{1}{\sqrt{\Delta_3}}\frac{P^2[A\,E]\langle E\,B\rangle(\langle D|QP|E\rangle - (P.Q + \frac{\sqrt{\Delta_3}}{2})\langle D\,E\rangle)(\langle C|QP|E\rangle - (P.Q + \frac{\sqrt{\Delta_3}}{2})\langle C\,E\rangle)}{[E|P|E\rangle\langle E|QP|E\rangle^2}, \\ =& -\frac{P^2[A\,E]\langle E\,B\rangle(\langle D|QP|E\rangle\langle C\,E\rangle + \langle C|QP|E\rangle\langle D\,E\rangle - 2P.Q\langle D\,E\rangle\langle C\,E\rangle)}{[E|P|E\rangle\langle E|QP|E\rangle^2}. \end{aligned} \tag{3.4.27}$$

The final term, $\mathcal{G}_{1,\gamma}^{0;x}$ can also be solved by H_1^0 functions, but requires more work to make explicitly rational,

$$\begin{aligned} G_{1,\gamma}^{0;x} =& \frac{1}{\Delta_3}(P^2[A|Q|B\rangle - (P.Q - \frac{\sqrt{\Delta_3}}{2})[A|P|B\rangle) \\ &\times \frac{(\langle D|QP|E\rangle - (P.Q - \frac{\sqrt{\Delta_3}}{2})\langle D\,E\rangle)(\langle C|QP|E\rangle - (P.Q - \frac{\sqrt{\Delta_3}}{2})\langle C\,E\rangle)}{\langle E|QP|E\rangle^2} \\ &- \frac{1}{\Delta_3}(-P^2[A|Q|B\rangle + (P.Q + \frac{\sqrt{\Delta_3}}{2})[A|P|B\rangle) \\ &\times \frac{(\langle D|QP|E\rangle - (P.Q + \frac{\sqrt{\Delta_3}}{2})\langle D\,E\rangle)(\langle C|QP|E\rangle - (P.Q + \frac{\sqrt{\Delta_3}}{2})\langle C\,E\rangle)}{\langle E|QP|E\rangle^2}. \end{aligned} \tag{3.4.28}$$

Note that the above expression is of the general form

$$\begin{aligned} G_{1,\gamma}^{0;x} =& (\mathcal{A} + \sqrt{\Delta_3}\mathcal{B})(\mathcal{C} + \sqrt{\Delta_3}\mathcal{D}) \\ &- (-\mathcal{A} + \sqrt{\Delta_3}\mathcal{B})(\mathcal{C} - \sqrt{\Delta_3}\mathcal{D}), \end{aligned} \tag{3.4.29}$$

where $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ are some complicated, but fully rational, terms dependent on the various spinors and momenta. We can then note that the above expression provides a fully rational result,

$$G^{0;x}_{1,\gamma} = (\mathcal{A}\mathcal{C} + \Delta_3 \mathcal{B}\mathcal{D}) \,. \tag{3.4.30}$$

From equation 3.4.28, we can identify $\mathcal{A} = (P^2[A|Q|B\rangle - P.Q[A|P|B\rangle)$ and $\mathcal{B} = (\frac{[A|P|B\rangle}{2})$. We can then construct $G^{0;x}_{1,\gamma}$ by simply multiplying out the rest of the expression to identify the fully rational part and the part multiplied by $\sqrt{\Delta_3}$. This yields the result

$$\begin{aligned}
&G^{0;x}_{1,\gamma} = \\
&\frac{(P^2[A|Q|B\rangle - P.Q[A|P|B\rangle)(\langle D|QP|E\rangle\langle C|QP|E\rangle - P.Q(\langle D|QP|E\rangle\langle C\,E\rangle + \langle C|QP|E\rangle\langle D\,E\rangle))}{\Delta_3\langle E|QP|E\rangle^2} \\
&+\frac{(P^2[A|Q|B\rangle - P.Q[A|P|B\rangle)\langle C\,E\rangle\langle D\,E\rangle((P.Q)^2 + \frac{\Delta_3}{4})}{\Delta_3\langle E|QP|E\rangle^2} \\
&+\frac{[A|P|B\rangle((\langle D|QP|E\rangle\langle C\,E\rangle + \langle C|QP|E\rangle\langle D\,E\rangle) - 2P.Q\langle C\,E\rangle\langle D\,E\rangle)}{2\langle E|QP|E\rangle^2} \,.
\end{aligned} \tag{3.4.31}$$

The full, rational canonical form is thus

$$\begin{aligned}
&G^{0;x}_{1}(A,B,C,D,E,Q,P) = \frac{P^2[E\,A][E|P|B\rangle\langle D\,E\rangle\langle C\,E\rangle}{[E|P|E\rangle^2\langle E|QP|E\rangle} \\
&-\frac{P^2[A\,E]\langle E\,B\rangle((\langle D|QP|E\rangle\langle C\,E\rangle + \langle C|QP|E\rangle\langle D\,E\rangle - 2P.Q\langle D\,E\rangle\langle C\,E\rangle)}{[E|P|E\rangle\langle E|QP|E\rangle^2} \\
&+\frac{(P^2[A|Q|B\rangle - P.Q[A|P|B\rangle)(\langle D|QP|E\rangle\langle C|QP|E\rangle - P.Q(\langle D|QP|E\rangle\langle C\,E\rangle + \langle C|QP|E\rangle\langle D\,E\rangle))}{\Delta_3\langle E|QP|E\rangle^2} \\
&+\frac{(P^2[A|Q|B\rangle - P.Q[A|P|B\rangle)\langle C\,E\rangle\langle D\,E\rangle((P.Q)^2 + \frac{\Delta_3}{4})}{\Delta_3\langle E|QP|E\rangle^2} \\
&+\frac{[A|P|B\rangle((\langle D|QP|E\rangle\langle C\,E\rangle + \langle C|QP|E\rangle\langle D\,E\rangle) - 2P.Q\langle C\,E\rangle\langle D\,E\rangle)}{2\langle E|QP|E\rangle^2} \,.
\end{aligned} \tag{3.4.32}$$

The final canonical form we require at $O(l^0)$ is the $\mathcal{G}^{0;xx}_1$,

$$\mathcal{G}^{0;xx}_1(A,B,C,D,E,F,Q,l_1) = \frac{[A|l_1|B\rangle\langle l_1\,C\rangle\langle l_1\,D\rangle\langle l_1\,E\rangle}{(l_1+Q)^2\langle l_1\,F\rangle^3} \,. \tag{3.4.33}$$

Applying equation (3.4.2) and substituting in $\hat{P}$ and $\hat{Q}$, we obtain

$$\mathcal{G}^{0;xx}_1 = \frac{-P^2}{4\sqrt{\Delta_3}[\hat{P}\,\hat{Q}]}\frac{[A|P|l_1\rangle\langle B\,l_1\rangle\langle C\,l_1\rangle\langle D\,l_1\rangle\langle E\,l_1\rangle}{\langle \hat{P}\,l_1\rangle\langle \hat{Q}\,l_1\rangle\langle F\,l_1\rangle^3} \,. \tag{3.4.34}$$

Again we split the poles in this expression, attempting to preserve the symmetry between $\hat{P}$ and $\hat{Q}$,

$$
\begin{aligned}
\mathcal{G}_1^{0;xx} =& \frac{-P^2}{4\sqrt{\Delta_3}[\hat{P}\,\hat{Q}]} \frac{[A|P|l_1\rangle\langle B\,l_1\rangle\langle C\,l_1\rangle\langle D\,l_1\rangle}{\langle F\,l_1\rangle^3} \left(\frac{\langle E\,\hat{P}\rangle}{\langle \hat{P}\,l_1\rangle\langle \hat{Q}\,\hat{P}\rangle} + \frac{\langle E\,\hat{Q}\rangle}{\langle \hat{P}\,\hat{Q}\rangle\langle \hat{Q}\,l_1\rangle} \right), \\
=& \frac{-P^2}{4\sqrt{\Delta_3}[\hat{P}\,\hat{Q}]} \left(\frac{[A|P|l_1\rangle\langle B\,l_1\rangle\langle C\,l_1\rangle\langle D\,F\rangle}{\langle F\,l_1\rangle^3\langle \hat{Q}\,\hat{P}\rangle} \left(\frac{\langle E\,\hat{Q}\rangle}{\langle F\,\hat{Q}\rangle} - \frac{\langle E\,\hat{P}\rangle}{\langle F\,\hat{P}\rangle} \right) \right. \\
& - \frac{[A|P|l_1\rangle\langle B\,l_1\rangle\langle C\,F\rangle}{\langle F\,l_1\rangle^2\langle \hat{Q}\,\hat{P}\rangle} \left(\frac{\langle E\,\hat{P}\rangle\langle D\,\hat{P}\rangle}{\langle F\,\hat{P}\rangle^2} - \frac{\langle E\,\hat{Q}\rangle\langle D\,\hat{Q}\rangle}{\langle F\,\hat{Q}\rangle^2} \right) \\
& + \frac{[A|P|l_1\rangle\langle B\,F\rangle}{\langle F\,l_1\rangle\langle \hat{Q}\,\hat{P}\rangle} \left(\frac{\langle E\,\hat{Q}\rangle\langle D\,\hat{Q}\rangle\langle C\,\hat{Q}\rangle}{\langle F\,\hat{Q}\rangle^3} - \frac{\langle E\,\hat{P}\rangle\langle D\,\hat{P}\rangle\langle C\,\hat{P}\rangle}{\langle F\,\hat{P}\rangle^3} \right) \\
& \left. + \frac{[A|P|l_1\rangle}{\langle \hat{Q}\,\hat{P}\rangle} \left(\frac{\langle E\,\hat{Q}\rangle\langle D\,\hat{Q}\rangle\langle C\,\hat{Q}\rangle\langle B\,\hat{Q}\rangle}{\langle F\,\hat{Q}\rangle^3\langle l_1\,\hat{Q}\rangle} - \frac{\langle E\,\hat{P}\rangle\langle D\,\hat{P}\rangle\langle C\,\hat{P}\rangle\langle B\,\hat{P}\rangle}{\langle F\,\hat{P}\rangle^3\langle l_1\,\hat{P}\rangle} \right) \right),
\end{aligned}
\tag{3.4.35}
$$

thus we now have 4 terms. The simplest is very similar to the $G_{1,\alpha}^{0;x}$ term, with the difference that we obtain an $H_1^{0;xx}$ canonical form instead of an $H_1^{0;x}$.

$$
\begin{aligned}
\mathcal{G}_{1,\alpha}^{0;xx} =& \frac{-P^2}{4\sqrt{\Delta_3}[\hat{P}\,\hat{Q}]} \frac{[A|P|l_1\rangle\langle B\,l_1\rangle\langle C\,l_1\rangle\langle D\,F\rangle}{\langle F\,l_1\rangle^3\langle \hat{Q}\,\hat{P}\rangle} \left(\frac{\langle E\,\hat{Q}\rangle}{\langle F\,\hat{Q}\rangle} - \frac{\langle E\,\hat{P}\rangle}{\langle F\,\hat{P}\rangle} \right), \\
=& \frac{-1}{\sqrt{\Delta_3}} \mathcal{H}_1^{0;xx}([A|P, B, C; F; l_1)\langle D\,F\rangle \left(\frac{\langle E\,\hat{Q}\rangle}{\langle F\,\hat{Q}\rangle} - \frac{\langle E\,\hat{P}\rangle}{\langle F\,\hat{Q}\rangle} \right), \\
G_{1,\alpha}^{0;xx} =& \frac{P^2[A\,F][F|P|B\rangle[F|P|C\rangle\langle D\,F\rangle\langle E\,F\rangle}{[F|P|F\rangle^3\langle F|QP|F\rangle}.
\end{aligned}
\tag{3.4.36}
$$

Similarly the $G_{1,\beta}^{0;xx}$ term is analogous to the $G_{1,\beta}^{0;x}$ term, with the presence of an $H_1^{0;x}$ function rather than an H_1^0 function,

$$
\begin{aligned}
\mathcal{G}_{1,\beta}^{0;xx} =& \frac{1}{\sqrt{\Delta_3}} \mathcal{H}_1^{0;x}([A|P, B; F; l_1)\langle C\,F\rangle \left(\frac{\langle E\,\hat{P}\rangle\langle D\,\hat{P}\rangle}{\langle F\,\hat{P}\rangle^2} - \frac{\langle E\,\hat{Q}\rangle\langle D\,\hat{Q}\rangle}{\langle F\,\hat{Q}\rangle^2} \right), \\
G_{1,\beta}^{0;xx} =& \frac{P^2[A\,F][F|P|B\rangle\langle C\,F\rangle(\langle E|QP|F\rangle\langle D\,F\rangle + \langle D|QP|F\rangle\langle E\,F\rangle - 2P.Q\langle E\,F\rangle\langle D\,F\rangle)}{[F|P|F\rangle^2\langle F|QP|F\rangle^2}.
\end{aligned}
\tag{3.4.37}
$$

The γ and δ parts, however, require more effort to obtain a fully rational expression. The $\mathcal{G}_{1,\gamma}^{0;xx}$ is given by

$$
\begin{aligned}
G^{0;xx}_{1,\gamma} =& \frac{P^2[A\,F]\langle B\,F\rangle}{\sqrt{\Delta_3}[F|P|F\rangle} \\
&\times\Bigg(((\langle E|QP|F\rangle - (P.Q - \frac{\sqrt{\Delta_3}}{2})\langle E\,F\rangle) \\
&\times\frac{(\langle D|QP|F\rangle - (P.Q - \frac{\sqrt{\Delta_3}}{2})\langle D\,F\rangle)(\langle C|QP|F\rangle - (P.Q - \frac{\sqrt{\Delta_3}}{2})\langle C\,F\rangle)}{\langle F|QP|F\rangle^3} \\
&-((\langle E|QP|F\rangle - (P.Q + \frac{\sqrt{\Delta_3}}{2})\langle E\,F\rangle) \\
&\times\frac{(\langle D|QP|F\rangle - (P.Q + \frac{\sqrt{\Delta_3}}{2})\langle D\,F\rangle)(\langle C|QP|F\rangle - (P.Q + \frac{\sqrt{\Delta_3}}{2})\langle C\,F\rangle)}{\langle F|QP|F\rangle^3}\Bigg), \\
=&\frac{P^2[A\,F]\langle B\,F\rangle}{\sqrt{\Delta_3}[F|P|F\rangle}\Bigg(\frac{\langle E|QP|F\rangle\langle D|QP|F\rangle\langle C|QP|F\rangle}{\langle F|QP|F\rangle^3} \\
&-(P.Q - \frac{\sqrt{\Delta_3}}{2}) \\
&\times\frac{(\langle E|QP|F\rangle\langle D|QP|F\rangle\langle C\,F\rangle + \langle E|QP|F\rangle\langle D\,F\rangle\langle C|QP|F\rangle + \langle E\,F\rangle\langle D|QP|F\rangle\langle C|QP|F\rangle)}{\langle F|QP|F\rangle^3} \\
&+((P.Q)^2 - P.Q\sqrt{\Delta_3} + \frac{\Delta_3}{4}) \\
&\times\frac{(\langle E|QP|F\rangle\langle D\,F\rangle\langle C\,F\rangle + \langle E\,F\rangle\langle D|QP|F\rangle\langle C\,F\rangle + \langle E\,F\rangle\langle D\,F\rangle\langle C|QP|F\rangle)}{\langle F|QP|F\rangle^3} \\
&-\frac{((P.Q)^3 - (P.Q)^2\frac{3\sqrt{\Delta_3}}{2} + P.Q\frac{3\Delta_3}{4} - \frac{\Delta_3\sqrt{\Delta_3}}{8})\langle E\,F\rangle\langle D\,F\rangle\langle C\,F\rangle}{\langle F|QP|F\rangle^3} - \{\text{irrational conjugate}\}\Bigg).
\end{aligned}
\tag{3.4.38}
$$

The irrational pieces in the above expression cancel and we are left only with terms which are rational; in the above expression, these can be identified as any term containing a factor of $\sqrt{\Delta_3}$ in the numerator, due to the additional factor of $\frac{1}{\sqrt{\Delta_3}}$ multiplying the whole expression. This leaves a term of

$$
\begin{aligned}
&G^{0;xx}_{1,\gamma} = \\
&P^2[A\,F]\langle B\,F\rangle \\
&\times\frac{(\langle E|QP|F\rangle\langle D|QP|F\rangle\langle C\,F\rangle + \langle E|QP|F\rangle\langle D\,F\rangle\langle C|QP|F\rangle + \langle E\,F\rangle\langle D|QP|F\rangle\langle C|QP|F\rangle)}{[F|P|F\rangle\langle F|QP|F\rangle^3} \\
&-\frac{2P^2(P.Q)[A\,F]\langle B\,F\rangle(\langle E|QP|F\rangle\langle D\,F\rangle\langle C\,F\rangle + \langle E\,F\rangle\langle D|QP|F\rangle\langle C\,F\rangle + \langle E\,F\rangle\langle D\,F\rangle\langle C|QP|F\rangle)}{[F|P|F\rangle\langle F|QP|F\rangle^3} \\
&+\frac{P^2(3(P.Q)^2 + \frac{\Delta_3}{4})[A\,F]\langle B\,F\rangle\langle C\,F\rangle\langle D\,F\rangle\langle E\,F\rangle}{[F|P|F\rangle\langle F|QP|F\rangle^3}.
\end{aligned}
\tag{3.4.39}
$$

The term $\mathcal{G}^{0;xx}_{1,\delta}$ is similar to $\mathcal{G}^{0;xx}_{1,\gamma}$, except that instead of a single overall factor dependent upon A and B we have the sum

$$(\mp[A|Q|B\rangle P^2 \pm (P.Q \pm \frac{\sqrt{\Delta_3}}{2})[A|P|B\rangle)\,, \tag{3.4.40}$$

where the sign differs between the $\hat{P}$ and $\hat{Q}$ contribution. We thus obtain an overall term of the form of equation (3.4.29), and we can therefore get the full $G^{0;xx}_{1,\delta}$ term as with the $G^{0;x}_{1,\gamma}$ case, by identifying the rational and irrational parts of the expression,

$$\begin{aligned}
G^{0;xx}_{1,\delta} = &\frac{P^2[A|Q|B\rangle - P.Q[A|P|B\rangle}{\Delta_3 P^2\langle F|QP|F\rangle^3}\,(2\langle C|QP|F\rangle\langle D|QP|F\rangle\langle E|QP|F\rangle \\
&-2P.Q(\langle E|QP|F\rangle\langle D|QP|F\rangle\langle C\,F\rangle + \langle E|QP|F\rangle\langle C|QP|F\rangle\langle D\,F\rangle + \langle C|QP|F\rangle\langle D|QP|F\rangle\langle E\,F\rangle) \\
&+2((P.Q)^2 + \frac{\Delta_3}{4})(\langle E|QP|F\rangle\langle D\,F\rangle\langle C\,F\rangle + \langle D|QP|F\rangle\langle E\,F\rangle\langle C\,F\rangle + \langle C|QP|F\rangle\langle D\,F\rangle\langle E\,F\rangle) \\
&- 2((P.Q)^3 + \frac{3}{4}P.Q\Delta_3)\langle E\,F\rangle\langle D\,F\rangle\langle C\,F\rangle\Big) \\
&+\frac{[A|P|B\rangle}{2P^2\langle F|QP|F\rangle^3} \\
&\times(\langle E|QP|F\rangle\langle D|QP|F\rangle\langle C\,F\rangle + \langle E|QP|F\rangle\langle C|QP|F\rangle\langle D\,F\rangle + \langle C|QP|F\rangle\langle D|QP|F\rangle\langle E\,F\rangle \\
&-2P.Q(\langle E|QP|F\rangle\langle D\,F\rangle\langle C\,F\rangle + \langle D|QP|F\rangle\langle E\,F\rangle\langle C\,F\rangle + \langle C|QP|F\rangle\langle D\,F\rangle\langle E\,F\rangle) \\
&+ (3(P.Q)^2 + \frac{\Delta_3}{4})\langle E\,F\rangle\langle D\,F\rangle\langle C\,F\rangle\Big)\,.
\end{aligned} \tag{3.4.41}$$

This is the last element of the canonical basis of G-functions of overall power of zero in the loop momentum which we will require, although others can be constructed if needed. We must now consider the substantially more difficult class of G-functions which are linear or quadratic in l. This can be seen by considering the identity (3.4.2); since the canonical form is of order l^1 or higher, the sub-leading term in l will now be of order l^0 or higher and thus cannot be neglected as it will necessarily contribute to the bubble coefficient. We therefore necessarily have roughly twice as many terms to compute as for the $O(l^0)$ G-functions, before we even consider the larger number of momenta typically present in the function and the fact that one of the terms must be constructed from lower-order G-functions rather than the simpler H-functions due to the presence of the propagator $(l_1 + Q)^2$ in the first term of equation (3.4.2).

Nonetheless, our established methods for computing G-functions through introducing null momenta $\hat{P}$ and $\hat{Q}$ in order to split poles and obtain an expression in terms of simpler canonical forms will prove to be applicable to higher-order G-functions, with some modifications. However, we will first consider the simplest possible linear G-function, the $\mathcal{G}^1_0$ canonical form,

$$\mathcal{G}_0^1(A,B,C,D,Q,l_1) = \frac{[A|l_1|B\rangle[C|l_1|D\rangle}{(l_1+Q)^2}\,. \tag{3.4.42}$$

We could solve this canonical form by the $\hat{P}\,\hat{Q}$ method, but perhaps surprisingly this particular G-function turns out to be amenable to the simpler ansatz method. This is primarily due to the simple denominator structure; the lack of spinor products in the denominator severely limits the variety of distinct denominator structures to choose from. Specifically, the denominator can consist only of discrete powers of the Gram determinant Δ_3.

We first attempt to construct the canonical form purely out of terms with a single factor of Δ_3 in the denominator. We have a limited range of structures which can appear in the numerator.

- Simple spinor strings, e.g. $[A|P|B\rangle$, $[A|Q|D\rangle$.
- Longer strings, made possible due to the presence of two non-null momenta in the problem, e.g. $[A|P[Q,P]|B\rangle$. We expect to obtain such terms in irrational-conjugate pairs from the usual method, so we tend to construct such terms including the $[P,Q]$ commutator when they appear.
- We cannot obtain terms containing spinor strings such as $\langle B\,D\rangle$ or $[A\,C]$ since they are antisymmetric under interchange of the spinors. However with two non-null momenta we can construct spinor strings such as $[A|[P,Q]|C]$, which is symmetric under interchange of A and C and thus does not vanish due to crossing symmetry.
- Kinematic factors with momentum weight but no spinor weight, such as Q^2 or $P.Q$.

We can anticipate that the G_0^1 function will consist of two distinct pieces; one with overall momentum weights P^1, Q^{-1}, and one with weights P^0, Q^0. This structure can be anticipated from equation (3.4.2), but is more explicitly visible by considering the special case

$$
\begin{aligned}
\sum_i \mathcal{G}_0^1(A,B,Q_i,Q_i,Q,l_1) =& \sum_i \frac{[A|l_1|B\rangle[Q_i|l_1|Q_i\rangle}{(l_1+Q)^2}\,, \\
=& \frac{2[A|l_1|B\rangle(l_1.Q)}{(l_1+Q)^2}\,, \\
=& [A|l_1|B\rangle - Q^2\frac{[A|l_1|B\rangle}{(l_1+Q)^2}\,, \\
=& \mathcal{H}_0^1(A,B,l_1) - Q^2\mathcal{G}_0^0(A,B,Q,l_1)\,.
\end{aligned} \tag{3.4.43}
$$

Since the evaluated canonical form must obey the same limit, we can conclude that the canonical form must contain two terms of differing momentum weight in Q and P.

By constructing all possible terms up to terms with a squared Gram determinant in the denominator, we can construct a candidate expression which can be verified to obey the above limit $(C,D)\to\sum Q_i$, and the equivalent limit in P, where we expect the behaviour

$$
\begin{aligned}
\frac{2[A|l_1|B\rangle(l_1.P)}{(l_1+Q)^2} =& -\frac{P^2[A|l_1|B\rangle}{(l_1+Q)^2}\,, \\
=& -P^2\mathcal{G}_0^0(A,B,Q,l_1)\,.
\end{aligned} \tag{3.4.44}
$$

The candidate expression which obeys these limits correctly in addition to the other required properties is given by

$$
G_0^1(A,B,C,D,Q,P) = \left.\begin{aligned}
&-\tfrac{1}{12}\tfrac{[A|P[P,Q]|B\rangle[C|P|D\rangle}{\Delta_3} + \text{Permutations}(A,C;B,D) \\
&+\tfrac{P.Q}{6}\tfrac{[A|P[P,Q]|B\rangle[C|P[P,Q]|D\rangle}{\Delta_3^2} + \text{Permutations}(B,D) \\
&-\tfrac{P^2}{3}\tfrac{[A|P[P,Q]|B\rangle[C|Q[P,Q]|D\rangle}{\Delta_3^2} + \text{Permutations}(A,C;B,D) \\
&+\tfrac{P^2(P.Q)}{6}\tfrac{[A|Q[P,Q]|B\rangle[C|P|D\rangle}{\Delta_3^2} + \text{Permutations}(A,C;B,D) \\
&-\tfrac{P^2(P.Q)}{6}\tfrac{[A|P[P,Q]|B\rangle[C|Q|D\rangle}{\Delta_3^2} + \text{Permutations}(A,C;B,D) \\
&-\tfrac{P^2(P.Q)}{6}\tfrac{[A|[P,Q|C]\langle B|[P,Q]|D\rangle}{\Delta_3^2}
\end{aligned}\right\} O(Q^{-1})\text{ term} \tag{3.4.45}
$$

$$
\left.\begin{aligned}
&+\tfrac{1}{4}\tfrac{[A|[P,Q]|C]\langle B|[P,Q]|D\rangle}{\Delta_3} \\
&+\tfrac{1}{8}\tfrac{[A|P[P,Q]|B\rangle[C|Q|D\rangle}{\Delta_3} + \text{Permutations}(A,C;B,D) \\
&+\tfrac{1}{8}\tfrac{[A|Q[P,Q]|B\rangle[C|P|D\rangle}{\Delta_3} + \text{Permutations}(A,C;B,D) \\
&-\tfrac{P.Q}{2}\tfrac{[A|P[P,Q]|B\rangle[C|Q[P,Q]|D\rangle}{\Delta_3^2} + \text{Permutations}(A,C;B,D) \\
&-\tfrac{P^2}{4}\tfrac{[A|Q[P,Q]|B\rangle[C|Q[P,Q]|D\rangle}{\Delta_3^2} + \text{Permutations}(B,D) \\
&+\tfrac{Q^2}{4}\tfrac{[A|P[P,Q]|B\rangle[C|P[P,Q]|D\rangle}{\Delta_3^2} + \text{Permutations}(B,D) \\
&-\tfrac{Q^2(P.Q)}{4}\tfrac{[A|P[P,Q]|B\rangle[C|P|D\rangle}{\Delta_3^2} + \text{Permutations}(A,C;B,D) \\
&+\tfrac{P^2(P.Q)}{4}\tfrac{[A|Q[P,Q]|B\rangle[C|Q|D\rangle}{\Delta_3^2} + \text{Permutations}(A,C;B,D)
\end{aligned}\right\} O(Q^{0})\text{ term}\,. \tag{3.4.46}
$$

This provides a working expression for the G_0^1 function. However, the ansatz method rapidly becomes impractical due to the proliferation of possible terms for more complicated linear G-functions. For the $\mathcal{G}_1^1$ function, therefore, we use the more complicated, but more easily scaled recursive method. This begins of course by applying the identity (3.4.2),

$$\begin{aligned}\mathcal{G}_1^1(A,B,C,D,E,F,Q,l_1) =&\frac{[A|l_1|B\rangle[C|l_1|D\rangle\langle l_1\,E\rangle}{(l_1+Q)^2\langle l_1\,F\rangle}\,,\\ =&\frac{[A|P(P+Q)Q|l_1\rangle[C|l_1|D\rangle\langle l_1\,B\rangle\langle l_1\,E\rangle}{(l_1+Q)^2\langle l_1|PQ|l_1\rangle\langle l_1\,F\rangle}-\frac{[A|P|l_1\rangle[C|l_1|D\rangle\langle l_1\,B\rangle\langle l_1\,E\rangle}{\langle l_1|PQ|l_1\rangle\langle l_1\,F\rangle}\,.\end{aligned} \tag{3.4.47}$$

We must now consider how to evaluate the first term. This expression can be simplified considerably by introducing a non-null linear combination of P and Q, $\mathcal{Q}$, which we define as

$$\mathcal{Q}^\mu = Q^\mu + \frac{Q^2}{P^2}P^\mu\,. \tag{3.4.48}$$

Since the definition of $\mathcal{Q}$ is free of irrational terms, we typically leave our final expressions for canonical forms written in terms of $\mathcal{Q}$ rather than Q. We can substitute for Q in our expression for $\mathcal{G}_1^1$ using a table of replacement rules similar to that for $\hat{P}$ and $\hat{Q}$,

$\mathcal{Q}$-dependent object	Q-dependent object
$\Delta_3(P,\mathcal{Q})$	$\Delta_3(P,Q)$
$\mathcal{Q}^2$	$Q^2+2\frac{Q^2}{P^2}P.Q+\frac{Q^4}{P^2}$
$P.\mathcal{Q}$	$P.Q+Q^2$
$[A\|\mathcal{Q}\|B\rangle$	$[A\|Q\|B\rangle+\frac{Q^2}{P^2}[A\|P\|B\rangle$
$\langle A\|P\mathcal{Q}\|B\rangle$	$\langle A\|PQ\|B\rangle+Q^2\langle A\,B\rangle$
$\langle A\|\mathcal{Q}P\|B\rangle$	$\langle A\|QP\|B\rangle+Q^2\langle A\,B\rangle$
$[A\|P\mathcal{Q}\|B]$	$[A\|PQ\|B]+Q^2[A\,B]$
$[A\|\mathcal{Q}P\|B]$	$[A\|QP\|B]+Q^2[A\,B]$
$[A\|P\mathcal{Q}P\|B\rangle$	$[A\|PQP\|B\rangle+Q^2[A\|P\|B\rangle$
$[l_1\|\mathcal{Q}\|l_1\rangle$	$(l_1+Q)^2$

Figure 3.2: Conversion table between $\mathcal{Q}$ and Q dependent expressions

Using these replacement rules to substitute for Q, our expression for $\mathcal{G}_1^1$ becomes

$$\mathcal{G}_1^1=\frac{P^2[A|\mathcal{Q}|l_1\rangle[C|l_1|D\rangle\langle l_1\,B\rangle\langle l_1\,E\rangle}{[l_1|\mathcal{Q}|l_1\rangle\langle l_1|P\mathcal{Q}|l_1\rangle\langle l_1\,F\rangle}-\frac{[A|P|l_1\rangle[C|l_1|D\rangle\langle l_1\,B\rangle\langle l_1\,E\rangle}{\langle l_1|P\mathcal{Q}|l_1\rangle\langle l_1\,F\rangle}\,. \tag{3.4.49}$$

Since $\mathcal{Q}$ is simply a non-null momentum, we can proceed to introduce factors of $\hat{P}$

and $\hat{Q}$ as usual and split poles accordingly. For the first term, this is given by

$$
\begin{aligned}
&\frac{P^4[A|Q|l_1\rangle[C|l_1|D\rangle\langle l_1\,B\rangle\langle l_1\,E\rangle}{4\sqrt{\Delta_3}[l_1|Q|l_1\rangle[\hat{P}\,\hat{Q}]\langle l_1\,\hat{P}\rangle\langle l_1\,\hat{Q}\rangle\langle l_1\,F\rangle}\,,\\
=&\frac{P^4[A|Q|l_1\rangle[C|l_1|D\rangle\langle l_1\,B\rangle}{8\sqrt{\Delta_3}[l_1|Q|l_1\rangle[\hat{P}\,\hat{Q}]}\left(\frac{\langle F\,E\rangle}{\langle F\,\hat{P}\rangle\langle l_1\,F\rangle\langle l_1\,\hat{Q}\rangle}+\frac{\langle\hat{P}\,E\rangle}{\langle\hat{P}\,F\rangle\langle l_1\,\hat{P}\rangle\langle l_1\,\hat{Q}\rangle}\right.\\
&\left.+\frac{\langle F\,E\rangle}{\langle F\,\hat{Q}\rangle\langle l_1\,F\rangle\langle l_1\,\hat{P}\rangle}+\frac{\langle\hat{Q}\,F\rangle}{\langle\hat{Q}\,F\rangle\langle l_1\,\hat{P}\rangle\langle l_1\,\hat{Q}\rangle}\right)\,,\\
=&\frac{P^4[C|l_1|D\rangle\langle l_1\,B\rangle}{8\sqrt{\Delta_3}[l_1|Q|l_1\rangle[\hat{P}\,\hat{Q}]}\left(\frac{\langle E\,F\rangle}{\langle F\,\hat{P}\rangle}\left(\frac{[A|Q|F\rangle}{\langle F\,\hat{Q}\rangle\langle l_1\,F\rangle}+\frac{[A|Q|\hat{Q}\rangle}{\langle\hat{Q}\,F\rangle\langle l_1\,\hat{Q}\rangle}\right)\right.\\
&+\frac{\langle E\,\hat{P}\rangle}{\langle\hat{P}\,F\rangle}\left(\frac{[A|Q|\hat{P}\rangle}{\langle\hat{P}\,\hat{Q}\rangle\langle l_1\,\hat{P}\rangle}+\frac{[A|Q|\hat{Q}\rangle}{\langle\hat{Q}\,\hat{P}\rangle\langle l_1\,\hat{Q}\rangle}\right)\\
&\left.+\frac{\langle E\,F\rangle}{\langle F\,\hat{Q}\rangle}\left(\frac{[A|Q|F\rangle}{\langle F\,\hat{P}\rangle\langle l_1\,F\rangle}+\frac{[A|Q|\hat{P}\rangle}{\langle\hat{P}\,F\rangle\langle l_1\,\hat{P}\rangle}\right)+\frac{\langle E\,\hat{Q}\rangle}{\langle\hat{Q}\,F\rangle}\left(\frac{[A|Q|\hat{P}\rangle}{\langle\hat{P}\,\hat{Q}\rangle\langle l_1\,\hat{P}\rangle}+\frac{[A|Q|\hat{Q}\rangle}{\langle\hat{Q}\,\hat{P}\rangle\langle l_1\,\hat{Q}\rangle}\right)\right)\,.
\end{aligned}
\tag{3.4.50}
$$

We then collect terms for each pole in l_1, and pull out a common denominator factor of $\langle F|\hat{P}\hat{Q}|F\rangle$,

$$
\begin{aligned}
=&\frac{P^4[C|l_1|D\rangle\langle l_1\,B\rangle}{8\sqrt{\Delta_3}[l_1|Q|l_1\rangle\langle F|\hat{P}\hat{Q}|F\rangle}\left(-\frac{2[A|Q|F\rangle\langle F\,E\rangle}{\langle l_1\,F\rangle}\right.\\
&+\frac{1}{\langle l_1\,\hat{P}\rangle}\left([A|Q|\hat{P}\rangle\langle F\,E\rangle-\frac{[A|Q|\hat{P}\rangle\langle\hat{P}\,E\rangle\langle\hat{Q}\,F\rangle}{\langle\hat{P}\,\hat{Q}\rangle}-\frac{[A|Q|\hat{P}\rangle\langle\hat{Q}\,E\rangle\langle\hat{P}\,F\rangle}{\langle\hat{P}\,\hat{Q}\rangle}\right)\\
&\left.+\frac{1}{\langle l_1\,\hat{Q}\rangle}\left([A|Q|\hat{Q}\rangle\langle F\,E\rangle-\frac{[A|Q|\hat{Q}\rangle\langle\hat{P}\,E\rangle\langle\hat{Q}\,F\rangle}{\langle\hat{Q}\,\hat{P}\rangle}-\frac{[A|Q|\hat{Q}\rangle\langle\hat{Q}\,E\rangle\langle\hat{P}\,F\rangle}{\langle\hat{Q}\,\hat{P}\rangle}\right)\right)\,.
\end{aligned}
\tag{3.4.51}
$$

We can now use the Schouten identity to simplify the second two lines of this expression, for example the $\langle l_1\,\hat{P}\rangle$ pole,

$$
\begin{aligned}
&[A|Q|\hat{P}\rangle\langle F\,E\rangle-\frac{[A|Q|\hat{P}\rangle\langle\hat{P}\,E\rangle\langle\hat{Q}\,F\rangle}{\langle\hat{P}\,\hat{Q}\rangle}-\frac{[A|Q|\hat{P}\rangle\langle\hat{Q}\,E\rangle\langle\hat{P}\,F\rangle}{\langle\hat{P}\,\hat{Q}\rangle}\,,\\
=&\frac{[A|Q|\hat{P}\rangle(\langle\hat{Q}\,\hat{P}\rangle\langle F\,E\rangle+\langle\hat{Q}\,F\rangle\langle E\,\hat{P}\rangle)}{\langle\hat{P}\,\hat{Q}\rangle}-\frac{[A|Q|\hat{P}\rangle\langle\hat{P}\,E\rangle\langle\hat{Q}\,F\rangle}{\langle\hat{P}\,\hat{Q}\rangle}+[A|Q|\hat{P}\rangle\langle F\,E\rangle\,,\\
=&-\frac{8[A|Q|\hat{P}\rangle\langle E|\hat{P}\hat{Q}|F\rangle}{P^2}\,,
\end{aligned}
\tag{3.4.52}
$$

with a similar simplification for the $\langle l_1\,\hat{Q}\rangle$ pole, we obtain a full term of

$$\frac{P^2[C|l_1|D\rangle\langle l_1\,B\rangle}{[l_1|\mathcal{Q}|l_1\rangle\langle F|P\mathcal{Q}|F\rangle}\left(\frac{[A|\mathcal{Q}|F\rangle\langle F\,E\rangle}{\langle l_1\,F\rangle}+\frac{4}{P^2}\left(\frac{[A|\mathcal{Q}|\hat{P}\rangle\langle E|\hat{P}\hat{\mathcal{Q}}|F\rangle}{\langle l_1\,\hat{P}\rangle}+\frac{[A|\mathcal{Q}|\hat{\mathcal{Q}}\rangle\langle E|\hat{\mathcal{Q}}\hat{P}|F\rangle}{\langle l_1\,\hat{\mathcal{Q}}\rangle}\right)\right). \tag{3.4.53}$$

We can observe two things about the above calculation.

1. The calculation would proceed identically for the second term in $\mathcal{G}_1^1$, yielding a very similar term with the simple replacement of $[A|P|X\rangle$ for $[A|\mathcal{Q}|X\rangle$.
2. The calculation left the factor $[C|l_1|D\rangle$ untouched, implying that the calculation would proceed identically for any $O(l^1)$ or $O(l^2)$ term.

The second point in particular allows us to generalize the above calculation to the case of a general $\mathcal{G}_1^n$ function for $n \geq 1$,

$$\mathcal{G}_1^n = f^n(l_1)\frac{[A\,l_1]\langle B\,l_1\rangle\langle E\,l_1\rangle}{[l_1|\mathcal{Q}|l_1\rangle\langle F\,l_1\rangle}, \tag{3.4.54}$$

where $f^n(l_1)$ is a polynomial in l of order n. Applying the $\mathcal{G}_1^1$ calculation, we can infer the general expansion

$$\begin{aligned}\mathcal{G}_1^n =&\frac{P^2 f(l_1)\langle B\,l_1\rangle}{[l_1|\mathcal{Q}|l_1\rangle\langle F|P\mathcal{Q}|F\rangle}\left(\frac{[A|\mathcal{Q}|F\rangle\langle E\,F\rangle}{\langle F\,l_1\rangle}+\frac{4}{P^2}\left(\frac{[A|\mathcal{Q}|\hat{P}\rangle\langle E|\hat{P}\hat{\mathcal{Q}}|F\rangle}{\langle l_1\,\hat{P}\rangle}+\frac{[A|\mathcal{Q}|\hat{\mathcal{Q}}\rangle\langle E|\hat{\mathcal{Q}}\hat{P}|F\rangle}{\langle l_1\,\hat{\mathcal{Q}}\rangle}\right)\right)\\ &-\frac{f(l_1)\langle B\,l_1\rangle}{\langle F|P\mathcal{Q}|F\rangle}\left(\frac{[A|P|F\rangle\langle E\,F\rangle}{\langle F\,l_1\rangle}+\frac{4}{P^2}\left(\frac{[A|P|\hat{P}\rangle\langle E|\hat{P}\hat{\mathcal{Q}}|F\rangle}{\langle l_1\,\hat{P}\rangle}+\frac{[A|P|\hat{\mathcal{Q}}\rangle\langle E|\hat{\mathcal{Q}}\hat{P}|F\rangle}{\langle l_1\,\hat{\mathcal{Q}}\rangle}\right)\right).\end{aligned} \tag{3.4.55}$$

Applying this to the $\mathcal{G}_1^1$ case, we can now write the evaluated form in terms of known canonical forms,

$$\begin{aligned}G_1^1 = \;& -\tfrac{P^2[A|\mathcal{Q}|F\rangle\langle E\,F\rangle}{\langle F|P\mathcal{Q}|F\rangle}G_1^0(C,D,B,F,\mathcal{Q},P) && (G_{1,(1)}^1)\\ & +\tfrac{4[A|\mathcal{Q}|\hat{P}\rangle\langle E|\hat{P}\hat{\mathcal{Q}}|F\rangle}{\langle F|P\mathcal{Q}|F\rangle}G_1^{0;\hat{P}}(C,D,B,\mathcal{Q},P) && (G_{1,(2)}^1)\\ & +\tfrac{4[A|\mathcal{Q}|\hat{\mathcal{Q}}\rangle\langle E|\hat{\mathcal{Q}}\hat{P}|F\rangle}{\langle F|P\mathcal{Q}|F\rangle}G_1^{0;\hat{\mathcal{Q}}}(C,D,B,\mathcal{Q},P) && (G_{1,(3)}^1)\\ & -\tfrac{[A|P|F\rangle\langle E\,F\rangle}{\langle F|P\mathcal{Q}|F\rangle}H_1^1(C,D,B,F,P) && (G_{1,(4)}^1)\\ & -\tfrac{4[A|P|\hat{P}\rangle\langle E|\hat{P}\hat{\mathcal{Q}}|F\rangle}{P^2\langle F|P\mathcal{Q}|F\rangle}H_1^1(C,D,B,\hat{P},P) && (G_{1,(5)}^1)\\ & -\tfrac{4[A|P|\hat{\mathcal{Q}}\rangle\langle E|\hat{\mathcal{Q}}\hat{P}|F\rangle}{P^2\langle F|P\mathcal{Q}|F\rangle}H_1^1(C,D,B,\hat{\mathcal{Q}},P) && (G_{1,(6)}^1).\end{aligned} \tag{3.4.56}$$

The functions $G_1^{0;\hat{P}}$ and $G_1^{0;\hat{\mathcal{Q}}}$ denote G_1^0 functions with an $\langle l_1\,\hat{P}\rangle$ or $\langle l_1\,\hat{\mathcal{Q}}\rangle$ pole. In these special cases the standard G_0^1 canonical form becomes invalid due to containing a singular denominator factor $\langle\hat{P}|P\mathcal{Q}|\hat{P}\rangle = 0$. Therefore a different canonical form for these special cases must be found, given by

$$\mathcal{G}_1^{0;\{\hat{Q}/\hat{P}\}} = \frac{[A\,l_1]\langle B\,l_1\rangle\langle D\,l_1\rangle}{[l_1|Q|l_1\rangle\langle\{\hat{Q}/\hat{P}\}\,l_1\rangle}\,. \tag{3.4.57}$$

For the $\hat{P}$ case we can proceed similarly to the G_1^0, introducing $\hat{P}$ and $\hat{Q}$ and attempting to split poles as far as possible,

$$\begin{aligned}
\mathcal{G}_1^{0;\hat{P}} =& \frac{[A|P|l_1\rangle\langle B\,l_1\rangle\langle D\,l_1\rangle}{\langle l_1|PQ|l_1\rangle\langle\hat{P}\,l_1\rangle}\,,\\
=& \frac{P^2}{4\sqrt{\Delta_3}}\frac{[A|P|l_1\rangle\langle B\,l_1\rangle\langle D\,l_1\rangle}{[\hat{P}\,\hat{Q}]\langle\hat{Q}\,l_1\rangle\langle\hat{P}\,l_1\rangle^2}\,,\\
=& \frac{P^2[A|P|l_1\rangle\langle B\,l_1\rangle}{4\sqrt{\Delta_3}[\hat{P}\,\hat{Q}]\langle\hat{P}\,l_1\rangle}\left(\frac{\langle D\,\hat{P}\rangle}{\langle\hat{P}\,l_1\rangle\langle\hat{Q}\,\hat{P}\rangle} + \frac{\langle D\,\hat{Q}\rangle}{\langle\hat{P}\,\hat{Q}\rangle\langle\hat{Q}\,l_1\rangle}\right)\,,\\
=& \frac{\langle D\,\hat{P}\rangle[A|P|l_1\rangle\langle B\,l_1\rangle}{\sqrt{\Delta_3}\langle\hat{P}\,l_1\rangle^2} && \left(G_{1,\alpha}^{0;x}\right)\\
&-\frac{\langle D\,\hat{Q}\rangle}{\sqrt{\Delta_3}}\left(\frac{[A|P|l_1\rangle\langle B\,\hat{P}\rangle}{\langle\hat{P}\,l_1\rangle\langle\hat{Q}\,\hat{P}\rangle} + \frac{[A|P|l_1\rangle\langle B\,\hat{Q}\rangle}{\langle\hat{Q}\,l_1\rangle\langle\hat{P}\,\hat{Q}\rangle}\right) && \left(G_{1,\beta}^{0;x}\right)\,.
\end{aligned} \tag{3.4.58}$$

It is clear that if we were to continue splitting $\mathcal{G}_{1,\alpha}^{0;\hat{P}}$ as we did for $\mathcal{G}_1^0$, the result would diverge, as expected given the behaviour of G_1^0 with denominator spinor $\hat{P}$. Instead $\mathcal{G}_{1,\alpha}^{0;\hat{P}}$ must be evaluated using our result for $H_1^{0;x}$,

$$\begin{aligned}
G_{1,\alpha}^{0;\hat{P}} =& -\frac{\langle D\,\hat{P}\rangle[A|PP|\hat{P}][\hat{P}|P|B\rangle}{\sqrt{\Delta_3}[\hat{P}|P|\hat{P}\rangle^2}\,,\\
=& \frac{4[A|\hat{P}|D\rangle[\hat{P}|P|B\rangle}{\sqrt{\Delta_3}P^2}\,,\\
=& 2\frac{(P^2[A|Q|D\rangle - (P.Q - \frac{\sqrt{\Delta_3}}{2})[A|P|D\rangle)[\hat{P}|P|B\rangle}{\Delta_3 P^2}\,.
\end{aligned} \tag{3.4.59}$$

The $\mathcal{G}_{1,\beta}^{0;\hat{P}}$ part can be split into H_1^0 functions,

$$\begin{aligned}
G_{1,\beta}^{0;\hat{P}} =& -\frac{P^2\langle D\,\hat{Q}\rangle}{\sqrt{\Delta_3}\langle\hat{P}\,\hat{Q}\rangle}\left(\frac{[A|\hat{P}|B\rangle}{[\hat{P}|P|\hat{P}\rangle} - \frac{[A|\hat{Q}|B\rangle}{[\hat{Q}|P|\hat{Q}\rangle}\right)\,,\\
=& -\frac{8[\hat{P}|\hat{Q}|D\rangle}{\sqrt{\Delta_3}P^2}\left([A|\hat{P}|B\rangle - [A|\hat{Q}|B\rangle\right)\,,\\
=& \frac{4[\hat{P}|P|D\rangle}{\Delta_3 P^2}(P^2[A|Q|B\rangle - P.Q[A|P|B\rangle)\,,\\
=& 2\frac{[A|[Q,P]P|B\rangle[\hat{P}|P|D\rangle}{\Delta_3 P^2}\,,
\end{aligned} \tag{3.4.60}$$

where we have used the fact that

$$[A|\hat{Q}|B\rangle = -[A|\hat{P}|B\rangle + \frac{1}{2}[A|P|B\rangle\,, \tag{3.4.61}$$

and exploited the fact that since $\hat{P}$ is null, $[\hat{P}|\hat{P}|B\rangle = 0$. The full $G_1^{0;\hat{P}}$ is thus given by

$$G_1^{0;\hat{P}} = 2\frac{[A|[Q,P]P|B\rangle[\hat{P}|P|D\rangle + (P^2[A|Q|D\rangle - (P.Q - \frac{\sqrt{\Delta_3}}{2})[A|P|D\rangle)[\hat{P}|P|B\rangle}{P^2\Delta_3}\,. \tag{3.4.62}$$

The derivation for the $G_1^{0;\hat{Q}}$ proceeds almost identically, yielding an expression differing only by irrational conjugation,

$$G_1^{0;\hat{Q}} = 2\frac{[A|[Q,P]P|B\rangle[\hat{Q}|P|D\rangle + (P^2[A|Q|D\rangle - (P.Q + \frac{\sqrt{\Delta_3}}{2})[A|P|D\rangle)[\hat{Q}|P|B\rangle}{P^2\Delta_3}\,. \tag{3.4.63}$$

We therefore have all the canonical forms necessary to evaluate the G_1^1 function. We do, however, obtain an expression containing irrational terms. We therefore must consider how to systematically cancel these unphysical irrationalities and obtain an explicitly rational canonical form.

Examining equation (3.4.56), we can observe that two terms, $G^1_{1,(1)}$ and $G^1_{1,(4)}$, are explicitly rational and need no further manipulation. We therefore attempt to cancel the irrational parts between the two remaining pairs of terms, $G^1_{1,(2)}$ and $G^1_{1,(3)}$, and $G^1_{1,(5)}$ and $G^1_{1,(6)}$. The key to making this cancellation manifest is by examining the terms of $\langle E|\hat{P}\hat{Q}|F\rangle$ and $\langle E|\hat{Q}\hat{P}|F\rangle$ present in the opposing paired terms. From table (3.4) we obtain the expansion

$$\langle E|\hat{P}\hat{Q}|F\rangle = \frac{1}{4\Delta_3}\left(-P^2\sqrt{\Delta_3}\langle E|PQ|F\rangle + \left(\frac{P^2\Delta_3}{2} + \sqrt{\Delta_3}P^2(P.Q)\right)\langle E\,F\rangle\right)\,. \tag{3.4.64}$$

By absorbing the $P.Q$ into the spinor product in the second term using the Schouten identity, we obtain

$$\begin{aligned}\langle E|\hat{P}\hat{Q}|F\rangle =& \frac{1}{4\Delta_3}\left(-P^2\sqrt{\Delta_3}\langle E|PQ|F\rangle + \frac{P^2\sqrt{\Delta_3}}{2}(\langle E|PQ|F\rangle + \langle E|QP|F\rangle) + \frac{P^2\Delta_3}{2}\langle E\,F\rangle\right)\,,\\ =& \frac{1}{4\Delta_3}\left(\frac{P^2\sqrt{\Delta_3}}{2}(\langle E|QP|F\rangle - \langle E|PQ|F\rangle) + \frac{P^2\Delta_3}{2}\langle E\,F\rangle\right)\,.\end{aligned} \tag{3.4.65}$$

The equivalent expression for the $\langle E|\hat{Q}\hat{P}|F\rangle$ is

$$\langle E|\hat{Q}\hat{P}|F\rangle = \frac{1}{4\Delta_3}\left(-\frac{P^2\sqrt{\Delta_3}}{2}(\langle E|QP|F\rangle - \langle E|PQ|F\rangle) + \frac{P^2\Delta_3}{2}\langle E\,F\rangle\right)\,. \tag{3.4.66}$$

These expressions are irrational conjugates of one another, which means that if we can write the sum of the $\langle l_1\,\hat{P}\rangle$ and $\langle l_1\,\hat{Q}\rangle$ poles in the form

$$\left(\frac{P^2\sqrt{\Delta_3}}{2}(\langle E|\mathcal{Q}P|F\rangle - \langle E|P\mathcal{Q}|F\rangle) + \frac{P^2\Delta_3}{2}\langle E\,F\rangle\right)(\mathcal{A}+\sqrt{\Delta_3}\mathcal{B})$$
$$+\left(-\frac{P^2\sqrt{\Delta_3}}{2}(\langle E|\mathcal{Q}P|F\rangle - \langle E|P\mathcal{Q}|F\rangle) + \frac{P^2\Delta_3}{2}\langle E\,F\rangle\right)(\mathcal{A}-\sqrt{\Delta_3}\mathcal{B}), \tag{3.4.67}$$

the sum will yield an explicitly rational expression,

$$P^2\Delta_3\langle E\,F\rangle\mathcal{A} + P^2\Delta_3(\langle E|\mathcal{Q}P|F\rangle - \langle E|P\mathcal{Q}|F\rangle)\mathcal{B}. \tag{3.4.68}$$

The problem is thus simply one of expanding out the $\hat{P}$ and $\hat{\mathcal{Q}}$ dependent parts of G_1^1 and identifying the rational and irrational parts. Beginning with $G_{1,(2)}^1$, and applying the above expression for $G_1^{0;\hat{P}}$ we obtain

$$\begin{aligned}
G_{1,(2)}^1 =& \frac{4\langle E|\hat{\mathcal{Q}}\hat{P}|F\rangle}{\langle F|P\mathcal{Q}|F\rangle} \\
&\left(\frac{2[C|[\mathcal{Q},P]|B\rangle[A|\mathcal{Q}\hat{P}P|D\rangle + 2(P^2[C|\mathcal{Q}|D\rangle - (P.\mathcal{Q} - \frac{\sqrt{\Delta_3}}{2})[C|P|D\rangle)[C|\mathcal{Q}\hat{P}P|B\rangle}{P^2\Delta_3}\right), \\
=& \frac{4\langle E|\hat{\mathcal{Q}}\hat{P}|F\rangle}{\langle F|P\mathcal{Q}|F\rangle}\left(\frac{[C|[\mathcal{Q},P]P|B\rangle(P^2\mathcal{Q}^2[A|P|D\rangle - (P.\mathcal{Q} - \frac{\sqrt{\Delta_3}}{2})[A|\mathcal{Q}|D\rangle)}{P^2\Delta_3\sqrt{\Delta_3}}\right. \\
&+\left.\frac{(P^2[C|\mathcal{Q}|D\rangle - (P.\mathcal{Q} - \frac{\sqrt{\Delta_3}}{2})[C|P|D\rangle)(P^2\mathcal{Q}^2[A|P|B\rangle - (P.\mathcal{Q} - \frac{\sqrt{\Delta_3}}{2})P^2[A|\mathcal{Q}|B\rangle)}{P^2\Delta_3\sqrt{\Delta_3}}\right),
\end{aligned} \tag{3.4.69}$$

$$\begin{aligned}
=&\frac{4\langle E|\hat{\mathcal{Q}}\hat{P}|F\rangle}{\langle F|P\mathcal{Q}|F\rangle}\left(\frac{[C|[\mathcal{Q},P]P|B\rangle(\sqrt{\Delta_3}P^2\mathcal{Q}^2[A|P|D\rangle - (\sqrt{\Delta_3}P.\mathcal{Q} - \frac{\Delta_3}{2})[A|\mathcal{Q}|D\rangle)}{P^2\Delta_3^2}\right. \\
&+\frac{\sqrt{\Delta_3}P^4\mathcal{Q}^2[C|\mathcal{Q}|D\rangle[A|P|B\rangle - P^4(\sqrt{\Delta_3}P.\mathcal{Q} - \frac{\Delta_3}{2})[C|\mathcal{Q}|D\rangle[A|\mathcal{Q}|B\rangle}{P^2\Delta_3^2} \\
&+\frac{-P^2\mathcal{Q}^2(\sqrt{\Delta_3}P.\mathcal{Q} - \frac{\Delta_3}{2})[C|P|D\rangle[A|P|B\rangle}{P^2\Delta_3^2} \\
&+\left.\frac{P^2((P.\mathcal{Q})^2\sqrt{\Delta_3} - P.\mathcal{Q}\Delta_3 + \frac{\sqrt{\Delta_3}\Delta_3}{4})[C|P|D\rangle[A|\mathcal{Q}|B\rangle}{P^2\Delta_3^2}\right).
\end{aligned} \tag{3.4.70}$$

By identifying the relevant terms in the above expression we can therefore postulate the following explicitly rational expression for the sum of $G_{1,(2)}^1$ and $G_{1,(3)}^1$,

$$
\begin{aligned}
G^1_{1,(2)}+G^1_{1,(3)} = &-\frac{P^2\langle E\,F\rangle}{\langle F|PQ|F\rangle}\Bigg(\frac{[C|[Q,P]P|B\rangle[A|Q|D\rangle + P^2[C|Q|D\rangle[A|Q|B\rangle}{2\Delta_3}\\
&+\frac{Q^2[C|P|D\rangle[A|P|B\rangle - 2P.Q[C|P|D\rangle[A|Q|B\rangle}{2\Delta_3}\Bigg)\\
&+\frac{P^2(\langle E|QP|F\rangle - \langle E|PQ|F\rangle)}{\langle F|PQ|F\rangle}\Bigg(\frac{[C|[Q,P]P|B\rangle(P.Q[A|Q|D\rangle - Q^2[A|P|D\rangle)}{\Delta_3^2}\\
&+\frac{-P^2Q^2[C|Q|D\rangle[A|P|B\rangle - ((P.Q)^2+\frac{\Delta_3}{4})[C|P|D\rangle[A|Q|B\rangle + P^2(P.Q)[C|Q|D\rangle[A|Q|B\rangle}{\Delta_3^2}\\
&+\frac{Q^2(P.Q)[C|P|D\rangle[A|P|B\rangle}{\Delta_3^2}\Bigg).
\end{aligned}
\tag{3.4.71}
$$

The same cancellation between $\langle E|\hat{P}\hat{Q}|F\rangle$ and $\langle E|\hat{Q}\hat{P}|F\rangle$ is present in the sum $G^1_{1,(5)} + G^1_{1,(6)}$, so again we expand out $G^1_{1,(6)}$ in order to identify the rational and irrational parts. We thus obtain the rational expression,

$$
\begin{aligned}
G^1_{1,(5)} + G^1_{1,(6)} = &-\frac{\langle E\,F\rangle}{\langle F|PQ|F\rangle}\Bigg(\frac{[C|P|B\rangle[A|P|D\rangle}{8} + \frac{[C|P|D\rangle[A|P|B\rangle}{8}\\
&+\frac{P^2[A|PQP|D\rangle[C|Q|B\rangle - P.Q[A|PQP|D\rangle[C|P|B\rangle - P^2(P.Q)[C|Q|B\rangle[A|P|D\rangle}{4\Delta_3}\\
&+\frac{((P.Q)^2+\frac{\Delta_3}{4})[C|P|B\rangle[A|P|D\rangle}{4\Delta_3}\\
&+\frac{P^2[A|PQP|B\rangle[C|Q|D\rangle - P.Q[A|PQP|B\rangle[C|P|D\rangle - P^2(P.Q)[C|Q|D\rangle[A|P|B\rangle}{4\Delta_3}\\
&+\frac{((P.Q)^2+\frac{\Delta_3}{4})[C|P|D\rangle[A|P|B\rangle}{4\Delta_3}\Bigg)\\
&+\frac{\langle E|[P,Q]|F\rangle}{\langle F|PQ|F\rangle}\Bigg(\frac{[C|P|B\rangle[A|P[P,Q]|D\rangle}{4\Delta_3} + \frac{[C|P|D\rangle[A|P[P,Q]|B\rangle}{4\Delta_3}\\
&+\frac{[A|P[P,Q]|B\rangle[C|P|D\rangle + [C|P[Q,P]|B\rangle[A|P|D\rangle}{8\Delta_3}\\
&+\frac{[A|P[P,Q]|D\rangle[C|P|B\rangle + [C|P[Q,P]|D\rangle[A|P|B\rangle}{8\Delta_3}\Bigg).
\end{aligned}
\tag{3.4.72}
$$

The only other linear G-function which must be evaluated is the $\mathcal{G}^{1;x}_1$ canonical form,

$$
\mathcal{G}^{1;x}_1(A,B,E,F;C_1,C_2;D;Q,l_1) = \frac{[A|l_1|B\rangle[E|l_1|F\rangle\langle l_1\,C_1\rangle\langle l_1\,C_2\rangle}{(l_1+Q)^2\langle l_1\,D\rangle^2}. \tag{3.4.73}
$$

This canonical form can be split in a very similar fashion to $\mathcal{G}^1_1$, leaving us with an expression identical to equation (3.4.56) with the exception of a single additional factor of $\langle l_1\,D\rangle$ in the denominator,

$$
\begin{aligned}
G_1^{1;x} &= [E|l_1|F\rangle \frac{[A\,l_1]\langle l_1\,B\rangle\langle l_1\,C_1\rangle\langle l_1\,C_2\rangle}{[l_1|\mathcal{Q}|l_1\rangle\langle l_1\,D\rangle^2}\,,\\
&= \frac{P^2[E|l_1|F\rangle\langle l_1\,C_1\rangle\langle l_1\,C_2\rangle}{[l_1|\mathcal{Q}|l_1\rangle\langle D|P\mathcal{Q}|D\rangle}\left(\frac{[A|\mathcal{Q}|D\rangle\langle B\,D\rangle}{\langle D\,l_1\rangle^2}+\frac{4}{P^2}\left(\frac{[A|\mathcal{Q}|\hat{P}\rangle\langle B|\hat{P}\hat{\mathcal{Q}}|D\rangle}{\langle l_1\hat{P}\rangle\langle l_1\,D\rangle}+\frac{[A|\mathcal{Q}|\hat{\mathcal{Q}}\rangle\langle B|\hat{\mathcal{Q}}\hat{P}|D\rangle}{\langle l_1\hat{\mathcal{Q}}\rangle\langle l_1\,D\rangle}\right)\right)\\
&- \frac{[E|l_1|F\rangle\langle l_1\,C_1\rangle\langle l_1\,C_2\rangle}{\langle D|P\mathcal{Q}|D\rangle}\left(\frac{[A|P|D\rangle\langle B\,D\rangle}{\langle D\,l_1\rangle^2}+\frac{4}{P^2}\left(\frac{[A|P|\hat{P}\rangle\langle B|\hat{P}\hat{\mathcal{Q}}|D\rangle}{\langle l_1\hat{P}\rangle\langle l_1\,D\rangle}+\frac{[A|P|\hat{\mathcal{Q}}\rangle\langle B|\hat{\mathcal{Q}}\hat{P}|D\rangle}{\langle l_1\hat{\mathcal{Q}}\rangle\langle l_1\,D\rangle}\right)\right).
\end{aligned}
\tag{3.4.74}
$$

The first terms on each line can be identified as a $\mathcal{G}_1^{0;x}$ and an $\mathcal{H}_0^{1;x}$ function, respectively. The $\hat{P}$ and $\hat{\mathcal{Q}}$ dependent terms meanwhile require a single further splitting in order to reduce them to $\mathcal{G}_1^0$ and $\mathcal{H}_1^1$ functions,

$$
\begin{aligned}
\mathcal{G}_1^{1;x} =& \frac{P^2[E|l_1|F\rangle\langle l_1\,C_1\rangle}{[l_1|\mathcal{Q}|l_1\rangle\langle D|P\mathcal{Q}|D\rangle}\left(\frac{[A|\mathcal{Q}|D\rangle\langle B\,D\rangle\langle l_1\,C_2\rangle}{\langle D\,l_1\rangle^2}\right.\\
&+\frac{4}{P^2}\left(\frac{[A|\mathcal{Q}|\hat{P}\rangle\langle B|\hat{P}\hat{\mathcal{Q}}|D\rangle\langle\hat{P}\,C_2\rangle}{\langle l_1\hat{P}\rangle\langle\hat{P}\,D\rangle}+\frac{[A|\mathcal{Q}|\hat{P}\rangle\langle B|\hat{P}\hat{\mathcal{Q}}|D\rangle\langle D\,C_2\rangle}{\langle D\hat{P}\rangle\langle l_1\,D\rangle}\right.\\
&\left.\left.+\frac{[A|\mathcal{Q}|\hat{\mathcal{Q}}\rangle\langle B|\hat{\mathcal{Q}}\hat{P}|D\rangle\langle\hat{\mathcal{Q}}\,C_2\rangle}{\langle l_1\hat{\mathcal{Q}}\rangle\langle\hat{\mathcal{Q}}\,D\rangle}+\frac{[A|\mathcal{Q}|\hat{\mathcal{Q}}\rangle\langle B|\hat{\mathcal{Q}}\hat{P}|D\rangle\langle D\,C_2\rangle}{\langle D\hat{\mathcal{Q}}\rangle\langle l_1\,D\rangle}\right)\right)\\
&-\frac{[E|l_1|F\rangle\langle l_1\,C_1\rangle}{\langle D|P\mathcal{Q}|D\rangle}\left(\frac{[A|P|D\rangle\langle B\,D\rangle\langle l_1\,C_2\rangle}{\langle D\,l_1\rangle^2}\right.\\
&+\frac{4}{P^2}\left(\frac{[A|P|\hat{P}\rangle\langle B|\hat{P}\hat{\mathcal{Q}}|D\rangle\langle\hat{P}\,C_2\rangle}{\langle l_1\hat{P}\rangle\langle\hat{P}\,D\rangle}+\frac{[A|P|\hat{P}\rangle\langle B|\hat{P}\hat{\mathcal{Q}}|D\rangle\langle D\,C_2\rangle}{\langle D\hat{P}\rangle\langle l_1\,D\rangle}\right.\\
&\left.\left.+\frac{[A|P|\hat{\mathcal{Q}}\rangle\langle B|\hat{\mathcal{Q}}\hat{P}|D\rangle\langle\hat{\mathcal{Q}}\,C_2\rangle}{\langle l_1\hat{\mathcal{Q}}\rangle\langle\hat{\mathcal{Q}}\,D\rangle}+\frac{[A|P|\hat{\mathcal{Q}}\rangle\langle B|\hat{\mathcal{Q}}\hat{P}|D\rangle\langle D\,C_2\rangle}{\langle D\hat{\mathcal{Q}}\rangle\langle l_1\,D\rangle}\right)\right).
\end{aligned}
\tag{3.4.75}
$$

Since the resulting terms pair off against each other in the same manner as those in the $\mathcal{G}_1^1$ function, we can apply equation (3.4.29) to extract the rational sum by expanding out each term and identifying the rational and irrational part,

$$
\begin{aligned}
&G_1^{1;x} = && \\
&\tfrac{P^2[A|\mathcal{Q}|D\rangle\langle B\,D\rangle}{\langle D|P\mathcal{Q}|D\rangle}G_1^{0;x}(E,F,C_1,C_2,D,\mathcal{Q},P) && G^{1;x}_{1,(1)}\\
&+\tfrac{4[A|\mathcal{Q}|\hat{P}\rangle\langle B|\hat{P}\hat{\mathcal{Q}}|D\rangle}{\langle D|P\mathcal{Q}|D\rangle}\left(\tfrac{\langle\hat{P}\,C_2\rangle}{\langle\hat{P}\,D\rangle}G_1^0(E,F,C_1,\hat{P},\mathcal{Q},P)+\tfrac{\langle D\,C_2\rangle}{\langle D\,\hat{P}\rangle}G_1^0(E,F,C_1,D,\mathcal{Q},P)\right) && G^{1;x}_{1,(2a)}+G^{1;x}_{1,(2b)}\\
&+\tfrac{4[A|\mathcal{Q}|\hat{\mathcal{Q}}\rangle\langle B|\hat{\mathcal{Q}}\hat{P}|D\rangle}{\langle D|P\mathcal{Q}|D\rangle}\left(\tfrac{\langle\hat{\mathcal{Q}}\,C_2\rangle}{\langle\hat{\mathcal{Q}}\,D\rangle}G_1^0(E,F,C_1,\hat{\mathcal{Q}},\mathcal{Q},P)+\tfrac{\langle D\,C_2\rangle}{\langle D\,\hat{\mathcal{Q}}\rangle}G_1^0(E,F,C_1,D,\mathcal{Q},P)\right) && G^{1;x}_{1,(3a)}+G^{1;x}_{1,(3b)}\\
&-\tfrac{[A|P|D\rangle\langle B\,D\rangle}{\langle D|P\mathcal{Q}|D\rangle}H_1^{1;x}(E,F,C_1,C_2,D,P) && G^{1;x}_{1,(4)}\\
&-\tfrac{4[A|P|\hat{P}\rangle\langle B|\hat{P}\hat{\mathcal{Q}}|D\rangle}{P^2\langle D|P\mathcal{Q}|D\rangle}\left(\tfrac{\langle\hat{P}\,C_2\rangle}{\langle\hat{P}\,D\rangle}H_1^1(E,F,C_1,\hat{P},P)+\tfrac{\langle D\,C_2\rangle}{\langle D\,\hat{P}\rangle}H_1^1(E,F,C_1,D,P)\right) && G^{1;x}_{1,(5a)}+G^{1;x}_{1,(5b)}\\
&-\tfrac{4[A|P|\hat{\mathcal{Q}}\rangle\langle B|\hat{\mathcal{Q}}\hat{P}|D\rangle}{P^2\langle D|P\mathcal{Q}|D\rangle}\left(\tfrac{\langle\hat{\mathcal{Q}}\,C_2\rangle}{\langle\hat{\mathcal{Q}}\,D\rangle}H_1^1(E,F,C_1,\hat{\mathcal{Q}},P)+\tfrac{\langle D\,C_2\rangle}{\langle D\,\hat{\mathcal{Q}}\rangle}H_1^1(E,F,C_1,D,P)\right) && G^{1;x}_{1,(6a)}+G^{1;x}_{1,(6b)}\,.
\end{aligned}
\tag{3.4.76}
$$

The b terms are the simplest sums, since the actual canonical forms are explicitly rational,

$$G^{1;x}_{1,(5b)} + G^{1;x}_{1,(6b)} = - \frac{\langle B\, D\rangle \langle D\, C_2\rangle [A|P[\mathcal{Q},P]|D\rangle}{\langle D|P\mathcal{Q}|D\rangle^2} H^1_1(E,F,C_1,D,P) + \frac{\langle B|[P,\mathcal{Q}]|D\rangle \langle D\, C_2\rangle [A|P|D\rangle}{\langle D|P\mathcal{Q}|D\rangle^2} H^1_1(E,F,C_1,D,P)\,, \tag{3.4.77}$$

$$\begin{aligned} G^{1;x}_{1,(2b)} + G^{1;x}_{1,(3b)} =& \frac{([A|[P,\mathcal{Q}]\mathcal{Q}|D\rangle \langle B\, D\rangle - [A|\mathcal{Q}|D\rangle \langle B|[P,\mathcal{Q}]|D\rangle) P^2 \langle D\, C_2\rangle}{2\langle D|P\mathcal{Q}|D\rangle^2} \\ & \times \left(\frac{[E|P[\mathcal{Q},P]|D\rangle \langle F|[P,\mathcal{Q}]|C_1\rangle}{2\Delta_3 \langle D|P\mathcal{Q}|D\rangle} + \frac{[E|P|D\rangle ([D|P|F\rangle \langle D\, C_1\rangle + [D|P|C_1\rangle \langle D\, F\rangle)}{2[D|P|D\rangle \langle D|P\mathcal{Q}|D\rangle} \right) . \end{aligned} \tag{3.4.78}$$

The a terms give

$$\begin{aligned} & G^{1;x}_{1,(2a)} + G^{1;x}_{1,(3a)} = \frac{\langle B\, D\rangle}{\Delta_3 \langle D|P\mathcal{Q}|D\rangle^2} \times \\ & \Big(\frac{[E|[\mathcal{Q},P]P|F\rangle}{2} (P^2 \langle C_2|\mathcal{Q}P|D\rangle [A|\mathcal{Q}|C_1\rangle + P^2\mathcal{Q}^2 \langle C_2\, D\rangle [A|P|C_1\rangle - 2P^2 P.\mathcal{Q} \langle C_2\, D\rangle [A|\mathcal{Q}|C_1\rangle) \\ & - \frac{\langle C_2|\mathcal{Q}P|D\rangle}{2} (-P^4 [E|\mathcal{Q}|C_1\rangle [A|\mathcal{Q}|F\rangle - P^2\mathcal{Q}^2 [E|P|C_1\rangle [A|P|F\rangle + 2P^2 P.\mathcal{Q} [E|P|C_1\rangle [A|\mathcal{Q}|F\rangle) \\ & + \frac{\langle C_2\, D\rangle}{2} (P^4\mathcal{Q}^2 [E|\mathcal{Q}|C_1\rangle [A|P|F\rangle - 2P^4 P.\mathcal{Q} [E|\mathcal{Q}|C_1\rangle [A|\mathcal{Q}|F\rangle - 2P^2\mathcal{Q}^2 P.\mathcal{Q} [E|P|C_1\rangle [A|P|F\rangle \\ & + (3(P.\mathcal{Q})^2 + \frac{\Delta_3}{4}) P^2 [E|P|C_1\rangle [A|\mathcal{Q}|F\rangle) \Big) \\ & - \frac{\langle B|P\mathcal{Q}|D\rangle - \langle B|\mathcal{Q}P|D\rangle}{\Delta_3^2 \langle D|P\mathcal{Q}|D\rangle^2} \big([E|[\mathcal{Q},P]P|F\rangle (P^2\mathcal{Q}^2 \langle C_2|\mathcal{Q}P|D\rangle [A|P|C_1\rangle \\ & - P^2 P.\mathcal{Q} \langle C_2|\mathcal{Q}P|D\rangle [A|\mathcal{Q}|C_1\rangle - P^2\mathcal{Q}^2 P.\mathcal{Q} \langle C_2\, D\rangle [A|P|C_1\rangle + ((P.\mathcal{Q})^2 + \frac{\Delta_3}{4}) P^2 \langle C_2\, D\rangle [A|\mathcal{Q}|C_1\rangle) \\ & - (\langle C_2|\mathcal{Q}P|D\rangle (-P^4\mathcal{Q}^2 [E|\mathcal{Q}|C_1\rangle [A|P|F\rangle + P^4 P.\mathcal{Q} [E|\mathcal{Q}|C_1\rangle [A|\mathcal{Q}|F\rangle \\ & + P^2\mathcal{Q}^2 P.\mathcal{Q} [E|P|C_1\rangle [A|P|F\rangle - P^2((P.\mathcal{Q})^2 + \frac{\Delta_3}{4}) [E|P|C_1\rangle [A|\mathcal{Q}|F\rangle) \\ & - \langle C_2\, D\rangle (-P^4\mathcal{Q}^2 P.\mathcal{Q} [E|\mathcal{Q}|C_1\rangle [A|P|F\rangle + P^4((P.\mathcal{Q})^2 + \frac{\Delta_3}{4}) [E|\mathcal{Q}|C_1\rangle [A|\mathcal{Q}|F\rangle \\ & + P^2\mathcal{Q}^2((P.\mathcal{Q})^2 + \frac{\Delta_3}{4}) [E|P|C_1\rangle [A|P|F\rangle - P^2 P.\mathcal{Q}((P.\mathcal{Q})^2 + \frac{3\Delta_3}{4}) [E|P|C_1\rangle [A|\mathcal{Q}|F\rangle))\big) \,, \end{aligned} \tag{3.4.79}$$

and

$$
\begin{aligned}
G^{1;x}_{1,5a} + G^{1;x}_{1,6a} = &\frac{\langle B\,D\rangle}{4\langle D|PQ|D\rangle^2}\Bigg(\frac{\langle C_2\,D\rangle}{2}([A|PQP|C_1\rangle[E|P|F\rangle + [A|PQP|F\rangle[E|P|C_1\rangle)\\
&+\frac{\langle C_2|QP|D\rangle}{2}([E|P|F\rangle[A|P|C_1\rangle + [E|P|C_1\rangle[A|P|F\rangle)\\
&- P.Q\langle C_2\,D\rangle([E|P|F\rangle[A|P|C_1\rangle + [E|P|C_1\rangle[A|P|F\rangle)\\
&+\frac{1}{\Delta_3}(P^2(\langle C_2|QP|D\rangle - P^2P.Q\langle C_2\,D\rangle)([A|PQP|C_1\rangle[E|Q|F\rangle + [A|PQP|F\rangle[E|Q|C_1\rangle)\\
&- \langle C_2|QP|D\rangle P^2P.Q([A|P|C_1\rangle[E|Q|F\rangle + [A|P|F\rangle[E|Q|C_1\rangle)\\
&- \langle C_2|QP|D\rangle P.Q([A|PQP|C_1\rangle[E|P|F\rangle + [A|PQP|F\rangle[E|P|C_1\rangle)\\
&+ \langle C_2\,D\rangle((P.Q)^2 + \frac{\Delta_3}{4})P^2([A|P|C_1\rangle[E|Q|F\rangle + [A|P|F\rangle[E|Q|C_1\rangle)\\
&+ \langle C_2\,D\rangle((P.Q)^2 + \frac{\Delta_3}{4})([A|PQP|C_1\rangle[E|P|F\rangle + [A|PQP|F\rangle[E|P|C_1\rangle)\\
&+ (\langle C_2|QP|D\rangle((P.Q)^2+\frac{\Delta_3}{4}) - \langle C_2\,D\rangle P.Q((P.Q)^2+\frac{3\Delta_3}{4}))([A|P|C_1\rangle[E|P|F\rangle+[A|P|F\rangle[E|P|C_1\rangle))\Bigg)\\
&- \frac{\langle B|[P,Q]|D\rangle}{4\Delta_3\langle D|PQ|D\rangle^2}(\langle C_2|QP|D\rangle([A|PQP|C_1\rangle[E|P|F\rangle + [A|PQP|F\rangle[E|P|C_1\rangle)\\
&- P.Q\langle C_2\,D\rangle([A|PQP|C_1\rangle[E|P|F\rangle + [A|PQP|F\rangle[E|P|C_1\rangle)\\
&- P.Q\langle C_2|QP|D\rangle([E|P|F\rangle[A|P|C_1\rangle + [E|P|C_1\rangle[A|P|F\rangle)\\
&+ \langle C_2\,D\rangle((P.Q)^2 + \frac{\Delta_3}{4})([E|P|F\rangle[A|P|C_1\rangle + [E|P|C_1\rangle[A|P|F\rangle)\\
&+ \frac{P^2\langle C_2\,D\rangle}{2}([A|PQP|C_1\rangle[E|Q|F\rangle + [A|PQP|F\rangle[E|Q|C_1\rangle)\\
&+ \frac{P^2\langle C_2|QP|D\rangle}{2}([A|P|C_1\rangle[E|Q|F\rangle + [A|P|F\rangle[E|Q|C_1\rangle)\\
&+ \frac{\langle C_2|QP|D\rangle}{2}([A|PQP|C_1\rangle[E|P|F\rangle + [A|PQP|F\rangle[E|P|C_1\rangle)\\
&- P^2P.Q\langle C_2\,D\rangle([A|P|C_1\rangle[E|Q|F\rangle + [A|P|F\rangle[E|Q|C_1\rangle)\\
&- P.Q\langle C_2\,D\rangle([A|PQP|C_1\rangle[E|P|F\rangle + [A|PQP|F\rangle[E|P|C_1\rangle)\\
&- P.Q\langle C_2|QP|D\rangle([A|P|C_1\rangle[E|P|F\rangle + [A|P|F\rangle[E|P|C_1\rangle)\\
&+ \frac{\langle C_2\,D\rangle}{2}(3(P.Q)^2 + \frac{\Delta_3}{4})([A|P|C_1\rangle[E|P|F\rangle + [A|P|F\rangle[E|P|C_1\rangle)\,.
\end{aligned}
\tag{3.4.80}
$$

The last remaining canonical form to be evaluated is the quadratic G-function, $\mathcal{G}_1^2$,

$$
\mathcal{G}_1^2(A,B,C,D,E,F,I,J,Q,l_1) = \frac{[A|l_1|B\rangle}{(l_1+Q)^2}[C|l_1|D\rangle[E|l_1|F\rangle\frac{\langle l_1\,I\rangle}{\langle l_1\,J\rangle}\,. \tag{3.4.81}
$$

This can be evaluated using the same procedure as $\mathcal{G}_1^1$, yielding an expression almost identical to equation (3.4.56) apart from the presence of different component canonical forms,

$$
\begin{aligned}
G_1^2 = \quad & -\frac{P^2[C|\mathcal{Q}|J\rangle\langle D\,J\rangle}{\langle J|P\mathcal{Q}|J\rangle} G_1^1(A,B,E,F,I,J,\mathcal{Q},P) && (G^2_{1,(1)}) \\
& +\frac{4[C|\mathcal{Q}|\hat{P}\rangle\langle D|\hat{P}\hat{\mathcal{Q}}|J\rangle}{\langle J|P\mathcal{Q}|J\rangle} G_1^{1;\hat{P}}(A,B,E,F,I,\mathcal{Q},P) && (G^2_{1,(2)}) \\
& +\frac{4[C|\mathcal{Q}|\hat{\mathcal{Q}}\rangle\langle D|\hat{\mathcal{Q}}\hat{P}|J\rangle}{\langle J|P\mathcal{Q}|J\rangle} G_1^{1;\hat{\mathcal{Q}}}(A,B,E,F,I,\mathcal{Q},P) && (G^2_{1,(3)}) \\
& -\frac{[C|P|J\rangle\langle D\,J\rangle}{\langle J|P\mathcal{Q}|J\rangle} H_1^2(A,B,E,F,I,J,P) && (G^2_{1,(4)}) \\
& -\frac{4[C|P|\hat{P}\rangle\langle D|\hat{P}\hat{\mathcal{Q}}|J\rangle}{P^2\langle J|P\mathcal{Q}|J\rangle} H_1^2(A,B,E,F,I,\hat{P},P) && (G^2_{1,(5)}) \\
& -\frac{4[C|P|\hat{\mathcal{Q}}\rangle\langle D|\hat{\mathcal{Q}}\hat{P}|J\rangle}{P^2\langle J|P\mathcal{Q}|J\rangle} H_1^2(A,B,E,F,I,\hat{\mathcal{Q}},P) && (G^2_{1,(6)})\,.
\end{aligned}
\tag{3.4.82}
$$

The only previously unseen canonical form in this expression is the $G_1^{1;\hat{P}}$ and $G_1^{1;\hat{\mathcal{Q}}}$ special case. This is solved by

$$
\begin{aligned}
G_1^{1,\hat{\mathcal{Q}}} = & -\frac{8\sqrt{\Delta_3}}{P^2\Delta_3}\Big([A|\hat{P}|B\rangle[E|\hat{P}|F\rangle[\hat{\mathcal{Q}}|P|I\rangle - [A|\hat{\mathcal{Q}}|B\rangle[E|\hat{\mathcal{Q}}|F\rangle[\hat{\mathcal{Q}}|P|I\rangle \\
& - [A|\hat{\mathcal{Q}}|I\rangle[E|\hat{\mathcal{Q}}|F\rangle[\hat{\mathcal{Q}}|P|B\rangle - [E|\hat{\mathcal{Q}}|I\rangle[A|\hat{\mathcal{Q}}|B\rangle[\hat{\mathcal{Q}}|P|F\rangle\Big) \\
& - \frac{8P.\mathcal{Q}}{P^2\Delta_3}\Big(([A|\hat{P}|B\rangle - [A|\hat{\mathcal{Q}}|B\rangle)([E|\hat{P}|F\rangle - [E|\hat{\mathcal{Q}}|F\rangle)[\hat{\mathcal{Q}}|P|I\rangle \\
& - ([A|\hat{P}|F\rangle[E|\hat{\mathcal{Q}}|B\rangle + [A|\hat{\mathcal{Q}}|F\rangle[E|\hat{P}|B\rangle)[\hat{\mathcal{Q}}|P|I\rangle \\
& - [A|\hat{\mathcal{Q}}|I\rangle([E|\hat{P}|F\rangle - [E|\hat{\mathcal{Q}}|F\rangle)[\hat{\mathcal{Q}}|P|B\rangle \\
& - [E|\hat{\mathcal{Q}}|I\rangle([A|\hat{P}|B\rangle - [A|\hat{\mathcal{Q}}|B\rangle)[\hat{\mathcal{Q}}|P|F\rangle\Big)\,,
\end{aligned}
\tag{3.4.83}
$$

with the equivalent expression for $G_1^{1;\hat{P}}$ being found by making the replacements in the above expression of $\hat{P} \leftrightarrow \hat{\mathcal{Q}}$, $\sqrt{\Delta_3} \to -\sqrt{\Delta_3}$, i.e. taking the irrational conjugate. With all the required canonical forms found, it is thus now a straightforward, though tedious, process to expand out all of the various terms and identify the rational and irrational parts in order to construct the rational canonical form. The sum of $G^2_{1,2}$ and $G^2_{1,3}$ is given by

$$
G^2_{1,2} + G^2_{1,3} = G^2_{1,23\alpha} + G^2_{1,23\beta} + G^2_{1,23\gamma} + G^2_{1,23\delta}\,, \tag{3.4.84}
$$

where the terms are given by

$$
\begin{aligned}
G^2_{1,23\alpha} = & \frac{\langle D\, J\rangle}{\langle J|P\mathcal{Q}|J\rangle\Delta_3^2} \times (P^6\mathcal{Q}^2[A|\mathcal{Q}|B\rangle[E|\mathcal{Q}|F\rangle[C|P|I\rangle - P^6 P.\mathcal{Q}[A|\mathcal{Q}|B\rangle[E|\mathcal{Q}|F\rangle[C|\mathcal{Q}|I\rangle \\
& -(P^4\mathcal{Q}^2 P.\mathcal{Q}[C|P|I\rangle - P^4((P.\mathcal{Q})^2 - \frac{\Delta_3}{4})[C|\mathcal{Q}|I\rangle)([A|\mathcal{Q}|B\rangle[E|P|F\rangle + [A|P|B\rangle[E|\mathcal{Q}|F\rangle) \\
& +(P^2\mathcal{Q}^2(P.\mathcal{Q})^2 + P^2\mathcal{Q}^2\frac{\Delta_3}{4})[A|P|B\rangle[E|P|F\rangle[C|P|I\rangle - P^2((P.\mathcal{Q})^3 - P.\mathcal{Q}\frac{\Delta_3}{4})[A|P|B\rangle[E|P|F\rangle[C|\mathcal{Q}|I\rangle) \\
& +(-(P^6\mathcal{Q}^2[A|\mathcal{Q}|B\rangle[E|\mathcal{Q}|F\rangle[C|P|I\rangle - P^6(P.\mathcal{Q})[A|\mathcal{Q}|B\rangle[E|\mathcal{Q}|F\rangle[C|\mathcal{Q}|I\rangle \\
& -(P^4\mathcal{Q}^2(P.\mathcal{Q})[C|P|I\rangle - P^4((P.\mathcal{Q})^2 + \frac{\Delta_3}{4})[C|\mathcal{Q}|I\rangle)([A|\mathcal{Q}|B\rangle[E|P|F\rangle + [A|P|B\rangle[E|\mathcal{Q}|F\rangle) \\
& +P^2\mathcal{Q}^2((P.\mathcal{Q})^2 + \frac{\Delta_3}{4})[A|P|B\rangle[E|P|F\rangle[C|P|I\rangle - P^2((P.\mathcal{Q})^3 + \frac{3}{4}(P.\mathcal{Q})\Delta_3)[A|P|B\rangle[E|P|F\rangle[C|\mathcal{Q}|I\rangle) \\
& -(P^6\mathcal{Q}^2[A|\mathcal{Q}|I\rangle[E|\mathcal{Q}|F\rangle[C|P|B\rangle - P^6(P.\mathcal{Q})[A|\mathcal{Q}|I\rangle[E|\mathcal{Q}|F\rangle[C|\mathcal{Q}|B\rangle \\
& -(P^4\mathcal{Q}^2(P.\mathcal{Q})[C|P|B\rangle - P^4((P.\mathcal{Q})^2 + \frac{\Delta_3}{4})[C|\mathcal{Q}|B\rangle)([A|\mathcal{Q}|I\rangle[E|P|F\rangle + [A|P|I\rangle[E|\mathcal{Q}|F\rangle) \\
& +P^2\mathcal{Q}^2((P.\mathcal{Q})^2 + \frac{\Delta_3}{4})[A|P|I\rangle[E|P|F\rangle[C|P|B\rangle - P^2((P.\mathcal{Q})^3 + \frac{3}{4}(P.\mathcal{Q})\Delta_3)[A|P|I\rangle[E|P|F\rangle[C|\mathcal{Q}|B\rangle) \\
& -(P^6\mathcal{Q}^2[E|\mathcal{Q}|I\rangle[A|\mathcal{Q}|B\rangle[C|P|F\rangle - P^6(P.\mathcal{Q})[E|\mathcal{Q}|I\rangle[A|\mathcal{Q}|B\rangle[C|\mathcal{Q}|F\rangle \\
& -(P^4\mathcal{Q}^2(P.\mathcal{Q})[C|P|F\rangle - P^4((P.\mathcal{Q})^2 + \frac{\Delta_3}{4})[C|\mathcal{Q}|F\rangle)([E|\mathcal{Q}|I\rangle[A|P|B\rangle + [E|P|I\rangle[A|\mathcal{Q}|B\rangle) \\
& +P^2\mathcal{Q}^2((P.\mathcal{Q})^2 + \frac{\Delta_3}{4})[E|P|I\rangle[A|P|B\rangle[C|P|F\rangle - P^2((P.\mathcal{Q})^3 + \frac{3}{4}(P.\mathcal{Q})\Delta_3)[E|P|I\rangle[A|P|B\rangle[C|\mathcal{Q}|F\rangle))\,,
\end{aligned} \tag{3.4.85}
$$

$$
\begin{aligned}
G^2_{1,23\beta} = & \frac{\langle D|[\mathcal{Q},P]|J\rangle}{2\langle J|P\mathcal{Q}|J\rangle\Delta_3^2} \times \big(P^6[A|\mathcal{Q}|B\rangle[E|\mathcal{Q}|F\rangle[C|\mathcal{Q}|I\rangle \\
& - P^4\mathcal{Q}^2[C|P|I\rangle([A|\mathcal{Q}|B\rangle[E|P|F\rangle + [A|P|B\rangle[E|\mathcal{Q}|F\rangle) \\
& + 2P^2\mathcal{Q}^2 P.\mathcal{Q}[A|P|B\rangle[E|P|F\rangle[C|P|I\rangle - P^2((P.\mathcal{Q})^2 - \frac{\Delta_3}{4})[A|P|B\rangle[E|P|F\rangle[C|\mathcal{Q}|I\rangle \\
& - P^6[A|\mathcal{Q}|B\rangle[E|\mathcal{Q}|F\rangle[C|\mathcal{Q}|I\rangle \\
& - (P^4\mathcal{Q}^2[C|P|I\rangle - 2P^4(P.\mathcal{Q})[C|\mathcal{Q}|I\rangle)([A|\mathcal{Q}|B\rangle[E|P|F\rangle + [A|P|B\rangle[E|\mathcal{Q}|F\rangle) \\
& + 2P^2\mathcal{Q}^2(P.\mathcal{Q})[A|P|B\rangle[E|P|F\rangle[C|P|I\rangle - P^2(3(P.\mathcal{Q})^2 + \frac{\Delta_3}{4})[A|P|B\rangle[E|P|F\rangle[C|\mathcal{Q}|I\rangle \\
& - P^6[A|\mathcal{Q}|I\rangle[E|\mathcal{Q}|F\rangle[C|\mathcal{Q}|B\rangle \\
& + (-P^4\mathcal{Q}^2[C|P|B\rangle + 2P^4(P.\mathcal{Q})[C|\mathcal{Q}|B\rangle)([A|\mathcal{Q}|I\rangle[E|P|F\rangle + [A|P|I\rangle[E|\mathcal{Q}|F\rangle) \\
& + 2P^2\mathcal{Q}^2(P.\mathcal{Q})[A|P|I\rangle[E|P|F\rangle[C|P|B\rangle - P^2(3(P.\mathcal{Q})^2 + \frac{\Delta_3}{4})[A|P|I\rangle[E|P|F\rangle[C|\mathcal{Q}|B\rangle \\
& - P^6[E|\mathcal{Q}|I\rangle[A|\mathcal{Q}|B\rangle[C|\mathcal{Q}|F\rangle \\
& + (-P^4\mathcal{Q}^2[C|P|F\rangle + 2P^4(P.\mathcal{Q})[C|\mathcal{Q}|F\rangle)([E|\mathcal{Q}|I\rangle[A|P|B\rangle + [E|P|I\rangle[A|\mathcal{Q}|B\rangle) \\
& + 2P^2\mathcal{Q}^2(P.\mathcal{Q})[E|P|I\rangle[A|P|B\rangle[C|P|F\rangle - P^2(3(P.\mathcal{Q})^2 + \frac{\Delta_3}{4})[E|P|I\rangle[A|P|B\rangle[C|\mathcal{Q}|F\rangle\Big)\,,
\end{aligned} \tag{3.4.86}
$$

$$
\begin{aligned}
G^2_{1,23\gamma} = &-\frac{P.\mathcal{Q}\langle D\,J\rangle}{2\Delta_3^2\langle J|P\mathcal{Q}|J\rangle} \\
&\times \big((2P^2[A|\mathcal{Q}|B\rangle - 2(P.\mathcal{Q})[A|P|B\rangle)(2P^2[E|\mathcal{Q}|F\rangle - 2(P.\mathcal{Q})[E|P|F\rangle)P^2[C|\mathcal{Q}|I\rangle \\
&+ 2(P^4[A|\mathcal{Q}|F\rangle[E|\mathcal{Q}|B\rangle - P^2(P.\mathcal{Q})([A|\mathcal{Q}|F\rangle[E|P|B\rangle + [A|P|F\rangle[E|\mathcal{Q}|B\rangle) \\
&+ ((P.\mathcal{Q})^2 - \frac{\Delta_3}{4})[A|P|F\rangle[E|P|B\rangle)P^2[C|\mathcal{Q}|I\rangle \\
&+ (2P^2[E|\mathcal{Q}|F\rangle - 2(P.\mathcal{Q})[E|P|F\rangle)P^2(P^2[A|\mathcal{Q}|I\rangle[C|\mathcal{Q}|B\rangle + \mathcal{Q}^2[A|P|I\rangle[C|P|B\rangle \\
&- 2(P.\mathcal{Q})[A|P|I\rangle[C|\mathcal{Q}|B\rangle) \\
&+ (2P^2[A|\mathcal{Q}|B\rangle - 2(P.\mathcal{Q})[A|P|B\rangle)P^2(P^2[E|\mathcal{Q}|I\rangle[C|\mathcal{Q}|F\rangle + \mathcal{Q}^2[E|P|I\rangle[C|P|F\rangle \\
&- 2(P.\mathcal{Q})[E|P|I\rangle[C|\mathcal{Q}|F\rangle))\ ,
\end{aligned} \tag{3.4.87}
$$

$$
\begin{aligned}
&G^2_{1,23\delta} = \\
&\frac{P.\mathcal{Q}\langle D|[P,\mathcal{Q}]|J\rangle}{\Delta_3^3\langle J|P\mathcal{Q}|J\rangle} \\
&\times \big((2P^2[A|\mathcal{Q}|B\rangle - 2P.\mathcal{Q}[A|P|B\rangle)(2P^2[E|\mathcal{Q}|F\rangle - 2P.\mathcal{Q}[E|P|F\rangle)(P^2\mathcal{Q}^2[C|P|I\rangle - P^2P.\mathcal{Q}[C|\mathcal{Q}|I\rangle) \\
&+ 2(P^4[A|\mathcal{Q}|F\rangle[E|\mathcal{Q}|B\rangle - P^2P.\mathcal{Q}([A|\mathcal{Q}|F\rangle[E|P|B\rangle + [A|P|F\rangle[E|\mathcal{Q}|B\rangle) \\
&+ ((P.\mathcal{Q})^2 - \frac{\Delta_3}{4})[A|P|F\rangle[E|P|B\rangle)(P^2\mathcal{Q}^2[C|P|I\rangle - (P.\mathcal{Q})P^2[C|\mathcal{Q}|I\rangle) \\
&+ (2P^2[E|\mathcal{Q}|F\rangle - 2P.\mathcal{Q}[E|P|F\rangle)P^2(P^2\mathcal{Q}^2[A|\mathcal{Q}|I\rangle[C|P|B\rangle - P^2(P.\mathcal{Q})[A|\mathcal{Q}|I\rangle[C|\mathcal{Q}|B\rangle \\
&- \mathcal{Q}^2(P.\mathcal{Q})[A|P|I\rangle[C|P|B\rangle + ((P.\mathcal{Q})^2 + \frac{\Delta_3}{4})[A|P|I\rangle[C|\mathcal{Q}|B\rangle) \\
&+ (2P^2[A|\mathcal{Q}|B\rangle - 2P.\mathcal{Q}[A|P|B\rangle)P^2(P^2\mathcal{Q}^2[E|\mathcal{Q}|I\rangle[C|P|F\rangle - P^2(P.\mathcal{Q})[E|\mathcal{Q}|I\rangle[C|\mathcal{Q}|F\rangle \\
&- \mathcal{Q}^2(P.\mathcal{Q})[E|P|I\rangle[C|P|F\rangle + ((P.\mathcal{Q})^2 + \frac{\Delta_3}{4})[E|P|I\rangle[C|\mathcal{Q}|F\rangle)\Big)\ .
\end{aligned} \tag{3.4.88}
$$

The sum of $G^2_{1,5}$ and $G^2_{1,6}$ is equal to

$$
G^2_{1,5} + G^2_{1,6} = G^2_{1,56\alpha} + G^2_{1,56\beta} + G^2_{1,56\gamma} + G^2_{1,56\delta} + G^2_{1,56\rho} + G^2_{1,56\sigma}\ , \tag{3.4.89}
$$

where the various terms are given by

$$
\begin{aligned}
G^2_{1,56\alpha} = &-\frac{\langle D|[\mathcal{Q},P]|J\rangle}{18\Delta_3P^2\langle J|P\mathcal{Q}|J\rangle} \\
&\times \big(([A|P|B\rangle[E|P|F\rangle + [A|P|F\rangle[E|P|B\rangle)(P^2[C|P\mathcal{Q}P|I\rangle - (P.\mathcal{Q})P^2[C|P|I\rangle) \\
&+ ([A|P|I\rangle[E|P|F\rangle + [A|P|F\rangle[E|P|I\rangle)(P^2[C|P\mathcal{Q}P|B\rangle - (P.\mathcal{Q})P^2[C|P|B\rangle) \\
&+ ([A|P|B\rangle[E|P|I\rangle + [A|P|I\rangle[E|P|B\rangle)(P^2[C|P\mathcal{Q}P|F\rangle - (P.\mathcal{Q})P^2[C|P|F\rangle)\big)\ ,
\end{aligned} \tag{3.4.90}
$$

$$
\begin{aligned}
G^2_{1,56\beta} = &-\frac{\langle D\,J\rangle}{36P^2\langle J|P\mathcal{Q}|J\rangle} \times \big(([A|P|B\rangle[E|P|F\rangle + [A|P|F\rangle[E|P|B\rangle)P^2[C|P|I\rangle \\
&+ ([A|P|I\rangle[E|P|F\rangle + [A|P|F\rangle[E|P|I\rangle)P^2[C|P|B\rangle \\
&+ ([A|P|B\rangle[E|P|I\rangle + [A|P|I\rangle[E|P|B\rangle)P^2[C|P|F\rangle\big)\ ,
\end{aligned} \tag{3.4.91}
$$

$$
\begin{aligned}
G^2_{1,56\gamma} = &- \frac{\langle D\, J\rangle}{36\Delta_3 P^2 \langle J|PQ|J\rangle} \\
&\times \big(P^4[A|P|B\rangle[E|Q|F\rangle[C|PQP|I\rangle - P^4(P.Q)[A|P|B\rangle[E|Q|F\rangle[C|P|I\rangle \\
&\quad - P^2(P.Q)[A|P|B\rangle[E|P|F\rangle[C|PQP|I\rangle + P^2((P.Q)^2 + \frac{\Delta_3}{4})[A|P|B\rangle[E|P|F\rangle[C|P|I\rangle \\
&\quad + P^4[E|P|B\rangle[A|Q|F\rangle[C|PQP|I\rangle - P^4(P.Q)[E|P|B\rangle[A|Q|F\rangle[C|P|I\rangle \\
&\quad - P^2(P.Q)[E|P|B\rangle[A|P|F\rangle[C|PQP|I\rangle + P^2((P.Q)^2 + \frac{\Delta_3}{4})[E|P|B\rangle[A|P|F\rangle[C|P|I\rangle \\
&\quad + \text{Permutations}(B, F, I)\big)\ ,
\end{aligned} \tag{3.4.92}
$$

$$
\begin{aligned}
G^2_{1,56\delta} = &- \frac{\langle D|[Q,P]|J\rangle}{72\Delta_3 P^2 \langle J|PQ|J\rangle} \\
&\times (P^4[E|Q|F\rangle[C|P|I\rangle[A|P|B\rangle + P^2[A|P|B\rangle[E|P|F\rangle[C|PQP|I\rangle \\
&- 2P^2(P.Q)[A|P|B\rangle[E|P|F\rangle[C|P|I\rangle \\
&+ P^4[A|Q|F\rangle[C|P|I\rangle[E|P|B\rangle + P^2[E|P|B\rangle[A|P|F\rangle[C|PQP|I\rangle \\
&- 2P^2(P.Q)[E|P|B\rangle[A|P|F\rangle[C|P|I\rangle \\
&+ \text{Permutations}(B, F, I))\ ,
\end{aligned} \tag{3.4.93}
$$

$$
\begin{aligned}
G^2_{1,56\rho} = &- \frac{\langle D|[Q,P]|J\rangle}{18\Delta_3^2 P^2 \langle J|PQ|J\rangle} \\
&\times (P^6[A|Q|B\rangle[E|Q|F\rangle[C|PQP|I\rangle - P^6(P.Q)[A|Q|B\rangle[E|Q|F\rangle[C|P|I\rangle \\
&- (P^4(P.Q)[C|PQP|I\rangle - P^4((P.Q)^2 + \frac{\Delta_3}{4})[C|P|I\rangle)([A|P|B\rangle[E|Q|F\rangle + [A|Q|B\rangle[E|P|F\rangle) \\
&+ (P^2((P.Q)^2 + \frac{\Delta_3}{4})[C|PQP|I\rangle \\
&- P^2((P.Q)^3 + \frac{3P.Q\Delta_3}{4})[C|P|I\rangle)[A|P|B\rangle[E|P|F\rangle \\
&+ \text{Permutations}(B, F, I))\,,
\end{aligned} \tag{3.4.94}
$$

$$
\begin{aligned}
G^2_{1,56\sigma} = &- \frac{\langle D\, J\rangle}{36\Delta_3 P^2 \langle J|PQ|J\rangle} \times (P^6[A|Q|B\rangle[E|Q|F\rangle[C|P|I\rangle \\
&- (-P^4[C|PQP|I\rangle + 2(P.Q)P^4[C|P|I\rangle)([A|P|B\rangle[E|Q|F\rangle + [A|Q|B\rangle[E|P|F\rangle) \\
&+ (-2(P.Q)P^2[C|PQP|I\rangle + P^2(3(P.Q)^2 + \frac{\Delta_3}{4})[C|P|I\rangle)[A|P|B\rangle[E|P|F\rangle \\
&+ \text{Permutations}(B, F, I))\,.
\end{aligned} \tag{3.4.95}
$$

These expressions can be evaluated by numerically checking both the crossing symmetry and the limits reducing them to lower canonical forms. With the full quadratic G-function thus evaluated, we have the complete canonical basis required to compute the bubble coefficients of the 6-gluon scalar loop in Chapter 5.

3.5 Triple Cuts

The same principle applied here to compute the bubble contributions of double cuts in the form of general canonical forms can also be applied to obtain canonical forms for the triangle contributions of triple cuts, using the same approach: The simplest non-trivial canonical forms are first computed explicitly, and then used to derive the more complicated forms using integrand-level spinor identities. This was done previously [79] by solving for triangle contributions of the simplest triple cuts by direct parametrization, for triple cuts of gluon amplitudes containing $\mathcal{N} = 1$ multiplet and scalar loops. Thus by combining the results of this chapter to compute double cuts with the previously derived triple cut canonical basis, and by solving the quadruple cuts algebraically within the framework of generalized Unitarity to compute the box coefficients, we are equipped to solve large classes of cuts appearing in gluon amplitudes by identifying them with elements of the canonical basis. This will be demonstrated in chapters 4 and 5 through the computation of the cut-constructible parts of an $\mathcal{N} = 1$ multiplet loop amplitude and a scalar loop amplitude, respectively.

Chapter 4

The 7-gluon NMHV $\mathcal{N} = 1$ chiral multiplet loop contribution

4.1 Motivation and Overall Structure

As both a proof-of-concept of the canonical basis approach and in order to calculate a previously unknown amplitude in a physically interesting theory, we can now apply the canonical basis approach to the 7-gluon next-to-MHV (NMHV) one-loop amplitude with an $\mathcal{N} = 1$ chiral Supersymmetric multiplet circulating in the loop. This amplitude has not been previously calculated, largely due to the difficulty of calculating a non-trivial amplitude with so many external particles. Computing this amplitude however is nonetheless worthwhile both in order to permit progress in amplitude computations via on-shell recursion, and due to the presence of an $\mathcal{N} = 1$ chiral part contributing to the gluonic Yang-Mills loop amplitude via equation (2.2.6).

The seven point NMHV consists of 4 basic primitive amplitudes corresponding to the four non-cyclic helicity configurations,

$$\begin{aligned}
A :& A_7(1^-,2^-,3^+,4^-,5^+,6^+,7^+)\,,\\
B :& A_7(1^-,2^-,3^+,4^+,5^-,6^+,7^+)\,,\\
C :& A_7(1^-,2^+,3^-,4^+,5^-,6^+,7^+)\,,\\
D :& A_7(1^-,2^-,3^-,4^+,5^+,6^+,7^+)\,.
\end{aligned} \tag{4.1.1}$$

All other possible NMHV helicity configurations can be related to these by cyclic symmmetry or charge conjugation, which effectively allows one to flip the direction of the internal particle and thus the ordering of the external legs, at the cost of introducing a factor of $(-1)^n$ where n is the number of external particles. In particular this implies that the configuration

$$A_7(1^+,2^+,3^+,4^-,5^+,6^-,7^-)\,, \tag{4.1.2}$$

can be mapped onto configuration A by charge conjugation and relabelling. The simplest of these four amplitudes is amplitude D, due to it being a "split-helicity" amplitude. This class of amplitudes has been previously solved for the general case [22], and as such we need only calculate the three previously unknown amplitudes A, B and C.

Since we are calculating an $\mathcal{N} = 1$ amplitude, the cancellations within the loop multiplet cause any terms in the integral reduction below bubble integrals to cancel. Specifically this means that there are no rational coefficients, and the amplitude has the integral reduction,

$$A_n^{\mathcal{N}=1} = \sum_{i\in\mathcal{C}} a_i I_4^i + \sum_{j\in\mathcal{D}} b_j I_3^j + \sum_{k\in\mathcal{E}} c_k I_2^k \,. \tag{4.1.3}$$

Since the coefficients in the above completely define the amplitude, we can thus describe the amplitude as "cut-constructible". As discussed in Appendix (A) the similarities in the IR behaviour of the integral functions for the boxes and the 1- and 2-mass triangles allow us to absorb the latter into the former in the form of "truncated" boxes, which have a different integral function to normal boxes and contain all information for both boxes and 1- and 2-mass triangles, but whose coefficients can be computed identically to normal boxes,

$$A_n^{\mathcal{N}=1\text{ chiral}} = \sum_{i\in\mathcal{C}} a_i \mathcal{F}_4^i + \sum_{j\in\mathcal{D}} b_j I_3^{3m,j} + \sum_{k\in\mathcal{E}} c_k I_2^k \,. \tag{4.1.4}$$

The main consequence of this is that the 1- and 2-mass triangle coefficients do not need to be computed independently; only the 3-mass triangle coefficients need be computed via the triple cuts.

We can thus construct the integral basis for the three partial amplitudes to be calculated, with coefficients to be determined by generalized 4-dimensional Unitarity; the bubbles consisting of s- and t-cuts, the boxes meanwhile consisting of 1-, 2- and 3-mass boxes, and for all three types of cut, only those momentum channels which yield permitted helicity configurations at all points in the calculation yielding a non-zero coefficient. For the A amplitude, we obtain 8 boxes, 2 triangles and 10 bubbles,

$$\begin{aligned}
A_7^{\mathcal{N}=1\text{ chiral}}(1^-,2^-,3^+,4^-,5^+,6^+,7^+) = {}& a_1^A \mathcal{F}^{3m}_{6\{71\}\{23\}\{45\}} \\
& + a_2^A \mathcal{F}^{2m,h}_{1\{23\}\{456\}7} + a_3^A \mathcal{F}^{2m,h}_{3\{45\}\{671\}2} + a_4^A \mathcal{F}^{2m,h}_{3\{456\}\{71\}2} + a_5^A \mathcal{F}^{2m,h}_{5\{671\}\{23\}4} \\
& + a_6^A \mathcal{F}^{2m,e}_{3\{45\}6\{712\}} + a_7^A \mathcal{F}^{1m}_{234\{5671\}} + a_8^A \mathcal{F}^{1m}_{345\{6712\}} \\
& + b_1^A I^{3m}_{\{23\}\{45\}\{671\}} + b_2^A I^{3m}_{\{71\}\{23\}\{456\}} \\
& + c_1^A I_2(t_{123}) + c_2^A I_2(t_{234}) + c_3^A I_2(t_{345}) + c_4^A I_2(t_{456}) + c_6^A I_(t_{671}) \\
& + c_7^A I_2(t_{712}) \\
& + d_2^A I_2(s_{23}) + d_3^A I_2(s_{34}) + d_4^A I_2(s_{45}) + d_7^A I_2(s_{71}) \,,
\end{aligned} \tag{4.1.5}$$

where the subscript for the boxes and triangles denotes the structure of the cuts; specifically, a group of numbers in braces, e.g. $\{345\}$ denotes a massive corner with legs 3, 4 and 5, whilst the same string without enclosing braces would denote the momenta being in three adjacent massless corners (as in the 1-mass boxes).

The B amplitude, meanwhile, consists of 11 boxes, 3 triangles and 11 bubbles,

$$
\begin{aligned}
A_7^{\mathcal{N}=1\text{ chiral}}(1^-,2^-,3^+,4^+,5^-,6^+,7^+) &= a_1^B\mathcal{F}^{3m}_{6\{71\}\{23\}\{45\}} + a_2^B\mathcal{F}^{3m}_{4\{56\}\{71\}\{23\}} \\
&+ a_3^B\mathcal{F}^{2m,h}_{1\{23\}\{456\}7} + a_4^B\mathcal{F}^{2m,h}_{3\{45\}\{671\}2} + a_5^B\mathcal{F}^{2m,h}_{6\{71\}\{234\}5} + a_6^B\mathcal{F}^{2m,h}_{1\{234\}\{56\}7} \\
&+ a_7^B\mathcal{F}^{2m,h}_{5\{671\}\{23\}4} + a_8^B\mathcal{F}^{2m,h}_{3\{456\}\{71\}2} \\
&+ a_9^B\mathcal{F}^{2m,e}_{3\{45\}6\{712\}} + a_{10}^B\mathcal{F}^{2m,e}_{4\{56\}7\{123\}} + a_{11}^B\mathcal{F}^{1m}_{456\{7123\}} \\
&+ b_1^BI^{3m}_{\{23\}\{45\}\{671\}} + b_2^BI^{3m}_{\{71\}\{23\}\{456\}} + b_3^BI^{3m}_{\{56\}\{71\}\{234\}} \\
&+ c_1^BI_2(t_{123}) + c_2^BI_2(t_{234}) + c_3^BI_2(t_{345}) + c_4^BI_2(t_{456}) + c_5^BI_2(t_{567}) \\
&+ c_6^BI_2(t_{671}) + c_7^BI_2(t_{712}) \\
&+ d_2^BI_2(s_{23}) + d_4^BI_2(s_{45}) + d_5^BI_2(s_{56}) + d_7^BI_2(s_{71}),
\end{aligned}
\tag{4.1.6}
$$

and the C amplitude consists of 19 boxes, 5 triangles and 13 bubbles,

$$
\begin{aligned}
&A_7^{\mathcal{N}=1\text{ chiral}}(1^-,2^+,3^-,4^+,5^-,6^+,7^+) = a_1^C\mathcal{F}^{3m}_{6\{71\}\{23\}\{45\}} + a_2^C\mathcal{F}^{3m}_{7\{12\}\{34\}\{56\}} \\
&+ a_3^C\mathcal{F}^{2m,h}_{1\{23\}\{456\}7} + a_4^C\mathcal{F}^{2m,h}_{2\{34\}\{567\}1} + a_5^C\mathcal{F}^{2m,h}_{3\{45\}\{671\}2} + a_6^C\mathcal{F}^{2m,h}_{4\{56\}\{712\}3} + a_7^C\mathcal{F}^{2m,h}_{6\{71\}\{234\}5} \\
&+ a_8^C\mathcal{F}^{2m,h}_{1\{234\}\{56\}7} + a_9^C\mathcal{F}^{2m,h}_{3\{456\}\{71\}2} + a_{10}^C\mathcal{F}^{2m,h}_{4\{567\}\{12\}3} + a_{11}^C\mathcal{F}^{2m,h}_{5\{671\}\{23\}4} + a_{12}^C\mathcal{F}^{2m,h}_{6\{712\}\{34\}5} \\
&+ a_{13}^C\mathcal{F}^{2m,e}_{4\{56\}7\{123\}} + a_{14}^C\mathcal{F}^{2m,e}_{6\{71\}2\{345\}} \\
&+ a_{15}^C\mathcal{F}^{1m}_{123\{4567\}} + a_{16}^C\mathcal{F}^{1m}_{234\{5671\}} + a_{17}^C\mathcal{F}^{1m}_{345\{6712\}} + a_{18}^C\mathcal{F}^{1m}_{456\{7123\}} + a_{19}^C\mathcal{F}^{1m}_{712\{3456\}} \\
&+ b_1^CI^{3m}_{\{12\}\{34\}\{567\}} + b_2^CI^{3m}_{\{712\}\{34\}\{56\}} + b_3^CI^{3m}_{\{71\}\{23\}\{456\}} + b_4^CI^{3m}_{\{71\}\{234\}\{56\}} + b_5^CI^{3m}_{\{671\}\{23\}\{45\}} \\
&+ c_1^CI_2(t_{123}) + c_2^CI_2(t_{234}) + c_3^CI_2(t_{345}) + c_4^CI_2(t_{456}) + c_5^CI_2(t_{567}) \\
&+ c_6^CI_2(t_{671}) + c_7^CI_2(t_{712}) \\
&+ d_1^CI_2(s_{12}) + d_2^CI_2(s_{23}) + d_3^CI_2(s_{34}) + d_4^CI_2(s_{45}) + d_5^CI_2(s_{56}) + d_7^CI_2(s_{71}).
\end{aligned}
\tag{4.1.7}
$$

4.2 Bubble Coefficients

Although there are 34 different bubble coefficients present in the three amplitudes, we can simplify the calculation considerably by noting that there is a considerable amount of symmetry between many of the cuts. In particular, many diagrams have identical helicity structure in their NMHV trees, differing only by the positioning of the negative leg on the MHV tree side (or positive leg on an $\overline{\text{MHV}}$ tree side). Rather than calculating term by term we can therefore instead calculate the general cut for each choice of NMHV tree, keeping the choice of negative leg on the MHV side arbitrary. We denote these general cuts for the t-cuts as C-functions, and those for the s-cuts as D-functions.

4.2.1 t-cuts

The five possible C-functions are shown in figure 4.1.

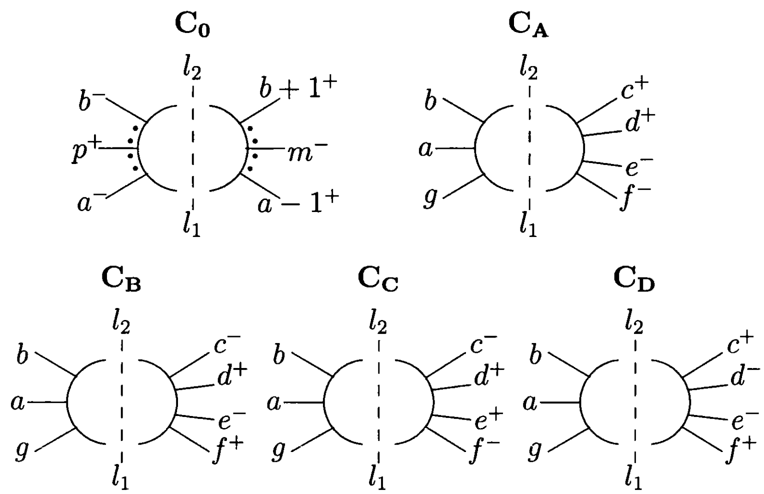


Figure 4.1: The 5 possible C-cuts. Note that C_0 denotes the general MHV$\times\overline{\text{MHV}}$ cut, as opposed to the specific 7-point case.

The simplest of these is the C_0 cut, which we can evaluate for the general all-n case, consisting as it does of simply an MHV tree times an $\overline{\text{MHV}}$ tree, where we solve for arbitrary choice of the negative leg on the MHV side, m, and the positive leg on the $\overline{\text{MHV}}$ side, p. The cut integrand is given by the product of the two trees,

$$C_0(m,p) = \sum_h \frac{\langle m\, l_1\rangle^2 \langle m\, l_2\rangle^2}{\langle a-1\, l_1\rangle \langle l_1\, l_2\rangle \langle l_2\, b+1\rangle \prod_{i=b+1}^{a-2} \langle i\, i+1\rangle} \frac{[p\, l_2]^2\, [p\, l_1]^2}{[l_1\, a]\, [b\, l_2]\, [l_2\, l_1] \prod i = a^{b-1}\, [j\, j+1]} \times \left(\frac{\langle m\, l_1\rangle\, [p\, l_1]}{\langle m\, l_2\rangle\, [p\, l_2]}\right)^h , \tag{4.2.1}$$

where h denotes the helicity of the particle circulating in the loop. For the $\mathcal{N}=1$ chiral multiplet this takes the value $h=0$ when l_2 is a complex scalar, and $h=\pm 1$ when l_2 is a fermion, as expected from table 2.2. This expression has overall momentum weight in the loop momentum of 2; however, we can write the supersymmetric contribution (consisting of the sum of h-dependent terms) in a form in which the expected cancellation of loop momentum power is manifest. The sum has the form

$$\rho^{SUSY} = \frac{A}{B} - 2 + \frac{B}{A}, \tag{4.2.2}$$

where in this case, $A = \langle m\, l_1\rangle\, [p\, l_1]$ and $B = \langle m\, l_2\rangle\, [p\, l_2]$. The key is to note that the sum is equivalent to

$$\rho^{SUSY} = \frac{(A-B)^2}{AB}\,. \tag{4.2.3}$$

In this case then we can write

$$\begin{aligned}\rho^{SUSY} =& \frac{(\langle m\, l_1\rangle\, [l_1\, p] - \langle m\, l_2\rangle\, [l_2\, p])^2}{\langle m\, l_1\rangle\, \langle m\, l_2\rangle\, [p\, l_1]\, [p\, l_2]}\,,\\ =& \frac{[p|P|m\rangle^2}{\langle m\, l_1\rangle\, \langle m\, l_2\rangle\, [p\, l_1]\, [p\, l_2]}\,.\end{aligned} \tag{4.2.4}$$

Thus, the cancellation within the multiplet has the effect of reducing the power of the loop momentum in the cut by two, to zero,

$$C_0(m,p) = \frac{[p|P|m\rangle^2}{P^2 \prod_{i=b+1}^{a-2} \langle i\, i+1\rangle \prod_{i=a}^{b-1} [j\, j+1]} \frac{\langle m\, l_1\rangle\, \langle m\, l_2\rangle\, [p\, l_1]\, [p\, l_2]}{\langle a-1\, l_1\rangle\, \langle b+1\, l_2\rangle\, [a\, l_1]\, [b\, l_2]}\,. \tag{4.2.5}$$

With an overall power of the loop momentum of zero, we can apply equation (3.2.8) and the simpler form of equation (3.2.9) to homogenize the cut in terms of purely l_1 angle spinor products,

$$C_0(m,p) = \frac{[p|P|m\rangle^2}{P^2 \prod_{i=b+1}^{a-2} \langle i\, i+1\rangle \prod_{i=a}^{b-1} [j\, j+1]} \frac{\langle m\, l_1\rangle^2\, [p|P|l_1\rangle^2}{\langle a-1\, l_1\rangle\, \langle b+1\, l_1\rangle\, [a|P|l_1\rangle [b|P|l_1\rangle}\,. \tag{4.2.6}$$

This cut is equivalent to the H_4 canonical form,

$$C_0(m,p) = \frac{[p|P|m\rangle^2}{P^2 \prod_{i=b+1}^{a-2} \langle i\, i+1\rangle \prod_{j=a}^{b-1} [j\, j+1]} H_4(m, m, [p|P, [p|P; a-1, b+1, [a|P, [b|P; P)\,. \tag{4.2.7}$$

The special case of this function which appears in the 7-point calculation is given by

$$\begin{aligned}C_0(a,b,c,d,e,f,g;p,m) =& \frac{[p|P_{gab}|m\rangle^2}{\langle c\, d\rangle\, \langle d\, e\rangle\, \langle e\, f\rangle\, [g\, a]\, [a\, b]\, P_{gab}^2}\\ &\times H_4(m, m, [p|P_{gab}, [p|P_{gab}; c, f, [b|P_{gab}, [g|P_{gab}; P_{gab})\,.\end{aligned} \tag{4.2.8}$$

The remaining C-cuts and D-cuts can be evaluated in the same manner, although they are more complicated in structure due to the presence of NMHV trees. This can be seen by explicitly evaluating the C_B function as follows. The cut in this case is given by

$$\sum_h A^{tree}(-l_1^h, 7, 1, 2, l_2^{-h}) \times A^{tree}(-l_2^h, 3^-, 4^+, 5^-, 6^+, l_1^{-h}) \,. \tag{4.2.9}$$

This gives rise to three terms due to the three terms in the NMHV 6-point tree amplitude [80]. For the case of a C_B cut in the channel t_{712}, the cut is a sum of three terms,

$$\begin{aligned}
C_B^1(h,t_{712}) =& \frac{[l_2|P_{456}|5\rangle^2 [l_1|P_{456}|5\rangle^2 \langle m_1\, l_1\rangle^2 \langle m_1\, l_2\rangle^2}{t_{456}[l_1\, l_2][l_2\, 3]\langle 4\, 5\rangle\langle 5\, 6\rangle [l_1|P_{456}|4\rangle [3|P_{456}|6\rangle \langle 7\, 1\rangle\langle 1\, 2\rangle\langle 2\, l_2\rangle\langle l_2\, l_1\rangle\langle l_1\, 7\rangle} \left(-\frac{[l_2|P_{456}|5\rangle}{[l_1|P_{456}|5\rangle}\right)^h , \\
C_B^2(h,t_{712}) =& \frac{[6|P_{56l_1}|3\rangle^2 \langle l_2\, 3\rangle^2 [6\, l_1]^2 \langle m_1\, l_1\rangle^2 \langle m_1\, l_2\rangle^2}{t_{56l_1}\langle l_2\, 3\rangle\langle 3\, 4\rangle [5\, 6][6\, l_1][5|P_{56l_1}|l_2\rangle [l_1|P_{56l_1}|4\rangle \langle 7\, 1\rangle\langle 1\, 2\rangle\langle 2\, l_2\rangle\langle l_2\, l_1\rangle\langle l_1\, 7\rangle} \left(-\frac{[6|P_{56l_1}|3\rangle}{\langle l_2\, 3\rangle\, [6\, l_1]}\right)^h , \\
C_B^3(h,t_{712}) =& \frac{[4|P_{345}|l_1\rangle^2 [4|P_{345}|l_2\rangle^2 \langle m_1\, l_1\rangle^2 \langle m_1\, l_2\rangle^2}{t_{345}\langle 6\, l_1\rangle\langle l_1\, l_2\rangle [3\, 4][4\, 5][3|P_{345}|6\rangle [5|P_{345}|l_2\rangle \langle 7\, 1\rangle\langle 1\, 2\rangle\langle 2\, l_2\rangle\langle l_2\, l_1\rangle\langle l_1\, 7\rangle} \left(-\frac{[4|P_{345}|l_1\rangle}{[4|P_{345}|l_2\rangle}\right)^h .
\end{aligned} \tag{4.2.10}$$

As before we can make the supersymmetric cancellations in the cuts explicit. For C_B^1 and C_B^3 this is simple, and the supersymmetric prefactor is given by

$$\begin{aligned}
\rho_1 =& \frac{\langle m_1|P_{712}P_{456}|5\rangle^2}{\langle m_1\, l_2\rangle\, [l_2|P_{456}|5\rangle\, \langle m_1\, l_1\rangle\, [l_1|P_{456}|5\rangle} , \\
\rho_3 =& \frac{[4|P_{345}|m_1\rangle^2 \langle l_1\, l_2\rangle^2}{\langle m_1\, l_1\rangle\, [4|P_{345}|l_1\rangle\, \langle m_1\, l_2\rangle\, [4|P_{345}|l_2\rangle} .
\end{aligned} \tag{4.2.11}$$

For C_B^2 the prefactor is given by

$$\rho_2 = \frac{([6\, 5]\,\langle 5\, 3\rangle\,\langle m_1\, l_2\rangle + [6\, l_1]\,\langle l_1\, 3\rangle\,\langle m_1\, l_2\rangle - \langle l_2\, 3\rangle\,[6\, l_1]\,\langle m_1\, l_1\rangle)^2}{[6|P_{56l_1}|3\rangle\,\langle l_2\, 3\rangle\,[6\, l_1]\,\langle m_1\, l_1\rangle\,\langle m_1\, l_2\rangle} . \tag{4.2.12}$$

To obtain the cancellation in this term we must apply the Schouten identity to the third term in the numerator,

$$\begin{aligned}
\rho_2 =& \frac{([6|P|l_2\rangle\,\langle m_1\, 3\rangle - [6\, 5]\,\langle 5\, 3\rangle\,\langle m_1\, l_2\rangle)^2}{[6|P_{56l_1}|3\rangle\,\langle l_2\, 3\rangle\,[6\, l_1]\,\langle m_1\, l_1\rangle\,\langle m_1\, l_2\rangle} , \\
=& \frac{\langle Y_{B2}\, l_2\rangle^2}{[6|P_{56l_1}|3\rangle\,\langle l_2\, 3\rangle\,[6\, l_1]\,\langle m_1\, l_1\rangle\,\langle m_1\, l_2\rangle} ,
\end{aligned} \tag{4.2.13}$$

where we have defined the spinor

$$|Y_{B2}\rangle = [6\, 5]\,\langle 5\, 3\rangle\,|m_1\rangle + [6\, 7]\,\langle m_1\, 3\rangle\,|7\rangle + [6\, 1]\,\langle m_1\, 3\rangle\,|1\rangle + [6\, 2]\,\langle m_1\, 3\rangle\,|2\rangle . \tag{4.2.14}$$

Applying these results we can reduce the cut to a sum of canonical forms, for example for the C_B^1 term,

$$
\begin{aligned}
C_B^1(t_{712}) =& \frac{\langle m_1\, l_1\rangle \, \langle m_1\, l_2\rangle}{\langle 7\,1\rangle\, \langle 1\,2\rangle \, \langle 2\, l_2\rangle \, \langle l_2\, l_1\rangle \, \langle l_1\, 7\rangle} \times \frac{[l_2|P_{456}|5\rangle [l_1|P_{456}|5\rangle \langle m_1|P_{712}P_{456}|5\rangle^2}{t_{456}\, [l_1\, l_2]\, [l_2\, 3]\, \langle 4\,5\rangle\, \langle 5\,6\rangle\, [l_1|P_{456}|4\rangle [3|P_{456}|6\rangle}\,, \\
=& \frac{\langle m_1|P_{712}P_{456}|5\rangle^2}{\langle 7\,1\rangle\, \langle 1\,2\rangle\, \langle 4\,5\rangle\, \langle 5\,6\rangle\, [3|P_{456}|6\rangle t_{456} t_{712}} \times \frac{\langle m_1\, l_1\rangle\, \langle m_1\, l_2\rangle}{\langle 2\, l_2\rangle\, \langle l_1\, 7\rangle} \times \frac{[l_2|P_{456}|5\rangle [l_1|P_{456}|5\rangle}{[l_2\, 3]\, [l_1|P_{456}|4\rangle}\,, \\
=& \frac{\langle m_1|P_{712}P_{456}|5\rangle^2}{\langle 7\,1\rangle \langle 1\,2\rangle \langle 4\,5\rangle \langle 5\,6\rangle [3|P_{456}|6\rangle t_{456} t_{712}} \times \frac{\langle m_1\, l_1\rangle \langle m_1\, l_2\rangle}{\langle 2\, l_2\rangle \langle l_1\, 7\rangle} \times \frac{\langle l_1|P_{712}P_{456}|5\rangle \langle l_2|P_{712}P_{456}|5\rangle}{\langle l_1|P_{712}|3] \langle l_2|P_{712}P_{456}|4\rangle}\,.
\end{aligned}
\tag{4.2.15}
$$

This is of the form of an H_4 function, so the contribution to the bubble coefficient is given by

$$
\begin{aligned}
C_B^1(t_{712}) \equiv& -\frac{\langle m_1|P_{712}P_{456}|5\rangle^2}{\langle 7\,1\rangle\, \langle 1\,2\rangle\, \langle 4\,5\rangle\, \langle 5\,6\rangle\, [3|P_{456}|6\rangle t_{456} t_{712}} \\
&\times H_4(m_1, m_1, P_{712}P_{456}|5\rangle, P_{712}P_{456}|5\rangle; 2, 7, P_{712}|3], P_{712}P_{456}|4\rangle; P_{712})\,.
\end{aligned}
\tag{4.2.16}
$$

Likewise the $C_B^2(t_{712})$ and $C_B^3(t_{712})$ give contributions

$$
\begin{aligned}
C_B^2(t_{712}) \equiv& \frac{1}{\langle 7\,1\rangle\, \langle 1\,2\rangle\, \langle 3\,4\rangle\, [5\,6]} G_4(6, 3; m_1, m_1, Y_{B2}, Y_{B2}; 2, 7, P_{34}|5], P_{712}P_{56}|4\rangle; P_{34}; P_{712})\,, \\
C_B^3(t_{712}) \equiv& \frac{[4|P_{456}|m_1\rangle^2}{\langle 7\,1\rangle\, \langle 1\,2\rangle\, [3\,4]\, [4\,5]\, [3|P_{345}|6\rangle t_{345}} H_4(m_1, m_1, P_{345}|4], P_{345}|4]; 2, 7, 6, P_{345}|5]; P_{712})\,.
\end{aligned}
\tag{4.2.17}
$$

From this we can define the general C_B cut,

$$
\begin{aligned}
C_B(a,b,c,d,e,f,g;m_1) \equiv& -\frac{\langle m_1|P_{gab}P_{def}|e\rangle^2}{\langle g\,a\rangle\, \langle a\,b\rangle\, \langle d\,e\rangle\, \langle e\,f\rangle\, [c|P_{def}|f\rangle t_{def} t_{gab}} \\
&\times H_4(m_1, m_1, P_{gab}P_{def}|e\rangle, P_{gab}P_{def}|e\rangle; b, g, P_{gab}|c], P_{gab}P_{def}|d\rangle; P_{gab}) \\
&+ \frac{1}{\langle g\,a\rangle\, \langle a\,b\rangle\, \langle c\,d\rangle\, [e\,f]} G_4(f, c; m_1, m_1, Y_{B2}, Y_{B2}; b, g, P_{cd}|e], P_{gab}P_{ef}|d\rangle; P_{cd}; P_{gab}) \\
&+ \frac{[d|P_{cde}|m_1\rangle^2}{\langle g\,a\rangle\, \langle a\,b\rangle\, [c\,d]\, [d\,e]\, [c|P_{cde}|f\rangle t_{cde}} H_4(m_1, m_1, P_{cde}|d], P_{cde}|d]; b, g, f, P_{cde}|e]; P_{gab})\,,
\end{aligned}
\tag{4.2.18}
$$

where the spinor $|Y_{B2}\rangle$ is defined as

$$
|Y_{B2}\rangle = [f\,e]\, \langle e\,c\rangle\, |m_1\rangle + [f\,g]\, \langle m_1\, c\rangle\, |g\rangle + [f\,a]\, \langle m_1\, c\rangle\, |a\rangle + [f\,b]\, \langle m_1\, c\rangle\, |b\rangle\,. \tag{4.2.19}
$$

Using the same approach one can solve the remaining C-functions,

$$
\begin{aligned}
&C_A(a,b,c,d,e,f,g;m_1) \equiv \\
&\quad \frac{\langle m_1|P_{gab}P_{cd}|e\rangle^2}{\langle c\,d\rangle\,\langle d\,e\rangle\,[f|P_{de}|c\rangle\,\langle g\,a\rangle\,\langle a\,b\rangle\,t_{cde}t_{gab}} H_3(m_1,m_1,P_{gab}P_{cd}|e\rangle;b,g,P_{gab}|f];P_{gab}) \\
&\quad -\frac{[d|P_{ef}|m_1\rangle^2}{\langle g\,a\rangle\,\langle a\,b\rangle\,[d\,e]\,[e\,f]\,[f|P_{de}|c\rangle t_{def}} H_3(m_1,m_1,P_{ef}|d];b,c,g;P_{gab})\,,
\end{aligned}
\tag{4.2.20}
$$

$$
\begin{aligned}
&C_C(a,b,c,d,e,f,g;m_1) \equiv \\
&\quad \frac{[d\,e]^3\,\langle m_1\,f\rangle^2}{t_{cde}\,[c\,d]\,[c|P_{de}|f\rangle\,\langle g\,a\rangle\,\langle a\,b\rangle} H_3(f,m_1,m_1;g,b,P_{cd}|e];P_{gab}) \\
&\quad -\frac{\langle m_1|P_{gab}P_{de}|f\rangle^2}{t_{def}t_{gab}\,\langle d\,e\rangle\,\langle e\,f\rangle\,[c|P_{de}|f\rangle\,\langle g\,a\rangle\,\langle a\,b\rangle} \\
&\quad \times \overline{H_4}(P_{de}|f\rangle,P_{de}|f\rangle,P_{gab}|m_1\rangle,P_{gab}|m_1\rangle;c,P_{ef}|d\rangle,P_{ga}|b\rangle,P_{ab}|g\rangle;P_{gab}) \\
&\quad +\frac{1}{[e\,f]\langle c\,d\rangle\langle g\,a\rangle\langle a\,b\rangle} G_5(e,c;m_1,m_1,P_{gab}|e],Y_C,Y_C;P_{cd}|e],P_{gab}P_{ef}|d\rangle,P_{gab}|f],b,g;P_{cd};P_{gab})\,,
\end{aligned}
\tag{4.2.21}
$$

$$
\begin{aligned}
&C_D(a,b,c,d,e,f,g;m_1) \equiv \\
&\quad \frac{\langle d\,e\rangle^3\,[f|P_{gab}|m_1\rangle^2}{t_{cde}t_{gab}\,\langle c\,d\rangle\,[f|P_{de}|c\rangle\,\langle g\,a\rangle\,\langle a\,b\rangle} H_3(m_1,m_1,P_{gab}|f];b,g,P_{gab}P_{cd}|e\rangle;P_{gab}) \\
&\quad -\frac{[f|P_{de}|m_1\rangle^2}{t_{def}\,[d\,e]\,[e\,f]\,[f|P_{de}|c\rangle\,\langle g\,a\rangle\,\langle a\,b\rangle} H_4(m_1,m_1,P_{de}|f],P_{de}|f];b,c,g,P_{ef}|d];P_{gab}) \\
&\quad -\frac{1}{\langle e\,f\rangle\,[c\,d]\,\langle g\,a\rangle\,\langle a\,b\rangle} G_5(c,e;e,m_1,m_1,Y_D,Y_D;f,b,g,P_{ef}|d],P_{gab}P_{cd}|e\rangle;P_{ef};P_{gab})\,,
\end{aligned}
\tag{4.2.22}
$$

where the spinors Y_C and Y_D are defined as

$$
\begin{aligned}
|Y_C\rangle &= [e\,f]\,\langle f\,c\rangle\,|m_1\rangle + \langle m_1\,c\rangle\,([e\,g]\,|g\rangle + [e\,a]\,|a\rangle + [e\,b]\,|b\rangle)\,, \\
|Y_D\rangle &= -\,[c\,d]\,\langle d\,e\rangle\,|m_1\rangle + \langle e\,m_1\rangle\,([c\,g]\,|g\rangle + [c\,a]\,|a\rangle + [c\,b]\,|b\rangle)\,.
\end{aligned}
\tag{4.2.23}
$$

With these general functions computed, we can solve for the various t-cut bubble coefficients in the three amplitudes,

$$
\begin{aligned}
c_1^A &= C_0(2,3,4,5,6,7,1;3,4),\; c_2^A = C_0(3,4,5,6,7,1,2;3,1), \\
c_3^A &= C_A(4,5,6,7,1,2,3;4),\; c_4^A = C_D(5,6,7,1,2,3,4;4), \\
c_6^A &= C_B(7,1,2,3,4,5,6;1),\; c_7^A = C_0(1,2,3,4,5,6,7;7,4),
\end{aligned}
\tag{4.2.24}
$$

$$
\begin{aligned}
c_1^B &= C_0(2,3,4,5,6,7,1;3,5),\; c_2^B = C_C(3,4,5,6,7,1,2;2), \\
c_3^B &= C_A(4,5,6,7,1,2,3;5),\; c_4^B = C_D(5,6,7,1,2,3,4;5), \\
c_5^B &= -C_A(6,5,4,3,2,1,7;5),\; c_6^B = -C_C(7,6,5,4,3,2,1;1), \\
&\qquad c_7^B = -\,C_0(1,7,6,5,4,3,2;7,5),
\end{aligned}
\tag{4.2.25}
$$

$$
\begin{gathered}
c_1^C = C_0(2,3,4,5,6,7,1;5,2),\ c_2^C = C_C(3,4,5,6,7,1,2;3),\\
c_3^C = C_0(4,5,6,7,1,2,3;4,1),\ c_4^C = -C_B(5,4,3,2,1,7,6;5),\\
c_5^C = C_B(6,7,1,2,3,4,5;5),\ c_6^C = -C_B(7,6,5,4,3,2,1;1),\\
c_7^C = C_B(1,2,3,4,5,6,7;1).
\end{gathered}
\tag{4.2.26}
$$

4.2.2 s-cuts

The six possible D-cuts are given in figure 4.2

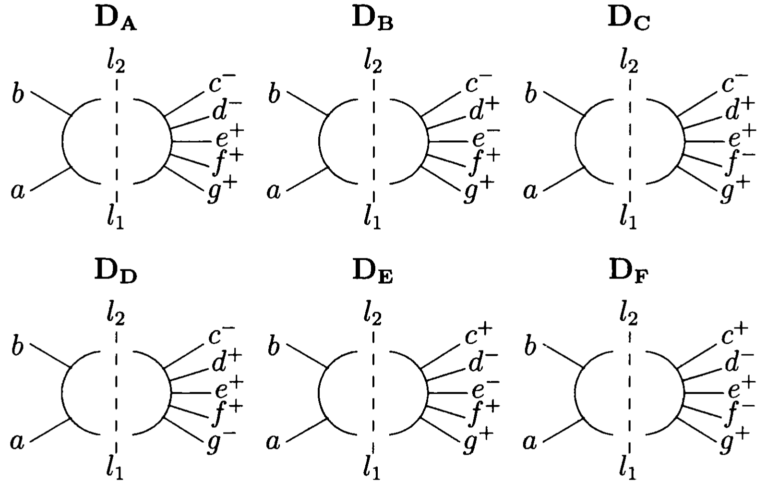


Figure 4.2: The 6 possible D-cuts

These can be computed using the 7-point NMHV tree amplitudes where two external particles are scalars or fermions, given in Appendix B. With these trees the D-functions can be computed in the same fashion as the C-functions. One obtains

$$
\begin{aligned}
&D_A(a,b,c,d,e,f,g) = \\
&\frac{[e|P_{cde}|a\rangle^2}{\langle a\,b\rangle\,\langle f\,g\rangle\,[c\,d]\,[d\,e]\,[c|P_{cde}|f\rangle t_{cde}} H_2(a, P_{cde}|e]; b, g; P_{ab}) \\
&-\frac{\langle f\,d\rangle}{[a\,b]\,\langle d\,e\rangle\,\langle e\,f\rangle\,\langle f\,g\rangle\,[c|P_{de}|f\rangle} \overline{H_4}(b, g, X_A, X_A; a, c, P_{cde}|f\rangle, P_{ab}P_{abc}P_{de}|f\rangle; P_{ab}) \\
&-\frac{1}{[a\,b]\langle d\,e\rangle\langle e\,f\rangle\langle f\,g\rangle[c|P_{de}|f\rangle} \overline{H_5}(b, P_{ab}|f\rangle, P_{ef}|d\rangle, X_A, X_A; a, P_{ab}|g\rangle, c, P_{cde}|f\rangle, P_{ab}P_{abc}P_{de}|f\rangle; P_{ab}) \\
&-\frac{\langle c\,d\rangle^4\,[g\,b]^2}{[a\,b]\,\langle c\,d\rangle\,\langle d\,e\rangle\,\langle e\,f\rangle\,[g|P_{ab}|c\rangle t_{gab}} H_2(a, P_{ab}|g]; b, P_{ab}P_{gab}|f\rangle; P_{ab}) \\
&-\frac{\langle a\,c\rangle^2\,[g|P_{abc}|d\rangle^4}{\langle a\,b\rangle\,\langle d\,e\rangle\,\langle e\,f\rangle\,[g|P_{ab}|c\rangle[g|P_{abc}|d\rangle t_{abc}t_{def}} H_2(a, c; b, P_{abc}P_{de}|f\rangle; P_{ab})\,,
\end{aligned}
\tag{4.2.27}
$$

where the spinor $|X_A]$ is defined as

$$
\begin{aligned}
|X_A] = &\langle f\, d\rangle\, [b\, a]\, \langle a\, g\rangle\, |g] + \langle f\, d\rangle\, [b\, a]\, \langle a\, f\rangle\, |f] - \langle a\, f\rangle\, [b\, a]\, \langle d\, e\rangle\, |e] \\
&+(\langle f\, g\rangle\, [g\, c]\, \langle c\, d\rangle + [c|P_{abc}|f\rangle\, \langle c\, d\rangle + \langle f\, d\rangle\, s_{ab})|b]\,.
\end{aligned}
\tag{4.2.28}
$$

The D_B function is given by

$$
\begin{aligned}
&D_B(a,b,c,d,e,f,g) = \\
&-\frac{1}{\langle e\, f\rangle\langle f\, g\rangle\langle a\, b\rangle[c\, d]} G_5(d,b;e,e,X_{B1a},X_{B1a},P_{ab}|d];a,g,P_{ab}P_{cd}|e\rangle,P_{ab}|c],P_{efg}|d];P_{efg};P_{ab}) \\
&+\frac{\langle b\, c\rangle^2}{\langle a\, b\rangle\langle f\, g\rangle[d\, e]} H_6(b,c,P_{fg}|d],P_{fg}|d],P_{fg}|d],P_{fg}|d];a,g,P_{fg}|e],P_{efg}|d],P_{fg}P_{de}|c\rangle,P_{defg}P_{de}|f\rangle;P_{ab}) \\
&-\frac{\langle c\, e\rangle^4\, [a\, f]}{\langle a\, b\rangle\langle c\, d\rangle\langle d\, e\rangle\langle f\, g\rangle t_{cde}} H_5(b,a,f,P_{fg}P_{cde}|b\rangle,P_{fg}P_{cde}|b\rangle;a,g,P_{ab}P_{cde}|f\rangle,P_{ab}P_{cde}|e\rangle,P_{fg}P_{cde}|c\rangle;P_{ab}) \\
&-\frac{\langle c\, e\rangle^4\, [b\, f]}{\langle a\, b\rangle\langle c\, d\rangle\langle d\, e\rangle\langle f\, g\rangle t_{cde}} H_5(b,b,f,P_{fg}P_{cde}|b\rangle,P_{fg}P_{cde}|b\rangle;a,g,P_{ab}P_{cde}|f\rangle,P_{ab}P_{cde}|e\rangle,P_{fg}P_{cde}|c\rangle;P_{ab}) \\
&-\frac{\langle c\, e\rangle^4\, [a\, g]}{\langle a\, b\rangle\langle c\, d\rangle\langle d\, e\rangle\langle f\, g\rangle t_{cde}} H_5(b,a,g,P_{fg}P_{cde}|b\rangle,P_{fg}P_{cde}|b\rangle;a,g,P_{ab}P_{cde}|f\rangle,P_{ab}P_{cde}|e\rangle,P_{fg}P_{cde}|c\rangle;P_{ab}) \\
&-\frac{\langle c\, e\rangle^4\, [b\, g]}{\langle a\, b\rangle\langle c\, d\rangle\langle d\, e\rangle\langle f\, g\rangle t_{cde}} H_5(b,b,g,P_{fg}P_{cde}|b\rangle,P_{fg}P_{cde}|b\rangle;a,g,P_{ab}P_{cde}|f\rangle,P_{ab}P_{cde}|e\rangle,P_{fg}P_{cde}|c\rangle;P_{ab}) \\
&+\frac{\langle c\, e\rangle^4\, [g\, a]^2\, \langle a\, b\rangle^2}{\langle c\, d\rangle\langle d\, e\rangle\langle e\, f\rangle\langle a\, b\rangle s_{ab}[g|P_{gab}|c\rangle t_{gab}} H_2(b,P_{ab}|g];a,P_{ab}P_{gab}|f\rangle;P_{ab}) \\
&-\frac{[g|P_{abc}|e\rangle^4\, \langle b\, c\rangle^2}{\langle a\, b\rangle\langle d\, e\rangle\langle e\, f\rangle[g|P_{abc}|c\rangle[g|P_{abc}|d\rangle t_{abc}t_{def}} H_2(b,c;a,P_{abc}P_{de}|f\rangle;P_{ab}) \\
&-\frac{[f\, g]^3\, \langle b\, c\rangle^2}{\langle a\, b\rangle\langle c\, d\rangle[e\, f][g|P_{ef}|d\rangle t_{efg}} H_2(b,c;a,P_{fg}|e];P_{ab})\,,
\end{aligned}
\tag{4.2.29}
$$

where the spinor $|X_{B1a}\rangle$ is defined as

$$
|X_{B1a}\rangle = [d\, a]\, \langle b\, e\rangle\, |a\rangle + [d|P_{bcd}|e\rangle|b\rangle\,. \tag{4.2.30}
$$

The function D_C is given by

$$
\begin{aligned}
&D_C(a,b,c,d,e,f,g) = \\
&\quad \frac{1}{[b\,c]\,\langle d\,e\rangle\,\langle e\,f\rangle\,\langle g\,a\rangle} G_5(c,a;e,e,X_{C1a},X_{C1a},P_{ga}|c];P_{ga}|b],f,P_{def}|c],P_{ga}P_{bc}|d\rangle,g;P_{bc};P_{ga}) \\
&\quad -\frac{[c\,d]^3\,\langle b\,a\rangle^2}{\langle e\,f\rangle\,\langle g\,a\rangle} H_6(a,b,e,e,e,e;f,g,P_{gab}P_{cd}|e\rangle,P_{ef}P_{cd}|b\rangle,P_{def}|d],P_{def}|c];P_{ga}) \\
&\quad -\frac{1}{\langle e\,f\rangle\,\langle b\,c\rangle\,\langle c\,d\rangle\,\langle g\,a\rangle\,t_{bcd}} \\
&\quad\quad \times G_5(P_{bcd}|b],a;e,e,X_{C1c},X_{C1c},P_{ga}P_{bcd}|b\rangle;f,g,P_{ef}P_{cd}|b\rangle,P_{ga}P_{bcd}|e\rangle,P_{ga}P_{bcd}|d\rangle;P_{ga}) \\
&\quad -\frac{\langle b\,e\rangle^4\,\langle a\,g\rangle^2\,[g\,f]^2}{\langle b\,c\rangle\,\langle c\,d\rangle\,\langle d\,e\rangle\,\langle g\,a\rangle\,[f|P_{cde}|b\rangle t_{fga}s_{ga}} \overline{H_2}(f,g;a,P_{bcd}|e\rangle;P_{ga}) \\
&\quad +\frac{[f|P_{cd}|e\rangle^4\,\langle b\,a\rangle^2}{\langle c\,d\rangle\,\langle d\,e\rangle\,[f|P_{de}|c\rangle[f|P_{ga}|b\rangle\,\langle g\,a\rangle\,t_{gab}t_{cde}} H_2(a,b;g,P_{gab}P_{cd}|e\rangle;P_{ga}) \\
&\quad -\frac{[f\,d]^4\,\langle b\,a\rangle^2}{\langle b\,c\rangle\,[d\,e]\,[e\,f]\,[f|P_{de}|c\rangle\,\langle g\,a\rangle\,t_{def}} H_2(a,b;g,P_{ef}|d];P_{ga})\,,
\end{aligned}
\tag{4.2.31}
$$

where the spinors $|X_{C1a}\rangle$ and $|X_{C1c}\rangle$ are defined as

$$
\begin{aligned}
|X_{C1a}\rangle &= -\,\langle e\,a\rangle\,([c\,g]\,|g\rangle + [c\,a]\,|a\rangle) + [c\,b]\,\langle b\,e\rangle\,|a\rangle\,, \\
|X_{C1c}\rangle &= \langle e\,b\rangle\,t_{bcd}|a\rangle + \langle e\,a\rangle\,([g|P_{cd}|b\rangle|g\rangle + [a|P_{cd}|b\rangle|a\rangle)\,.
\end{aligned}
\tag{4.2.32}
$$

The function D_D is given by

$$
\begin{aligned}
&D_D(a,b,c,d,e,f,g) = \\
&\quad \frac{1}{\langle a\,b\rangle\,\langle c\,d\rangle\,\langle d\,e\rangle\,[f\,g]} G_4(f,a;c,X_{D1},X_{D1},P_{ab}|f];b,P_{ab}|g],P_{ab}P_{fg}|e\rangle,P_{cde}|f];P_{cde};P_{ab}) \\
&\quad -\frac{\langle a\,g\rangle^2\,[f|P_{cde}|c\rangle^3}{\langle a\,b\rangle\,\langle c\,d\rangle\,\langle d\,e\rangle\,\langle e|P_{cd}P_{ab}|g\rangle t_{cde}t_{gab}} H_2(a,g;b,P_{cde}|f];P_{ab}) \\
&\quad -\frac{[d|P_{ef}|g\rangle^3\,\langle a\,g\rangle^2}{\langle a\,b\rangle\,\langle e\,f\rangle\,\langle f\,g\rangle\,[c\,d]\,[c|P_{ab}|g\rangle\langle g|P_{ab}P_{cd}|e\rangle} H_2(a,g;b,P_{cd}P_{ef}|g\rangle;P_{ab}) \\
&\quad +\frac{1}{\langle a\,b\rangle\,\langle e\,f\rangle\,\langle f\,g\rangle\,\langle c\,d\rangle\,t_{efg}} \\
&\quad\quad \times G_4(P_{ef}|g\rangle,a;c,P_{ab}P_{ef}|g\rangle,X_{D4},X_{D4};b,P_{ab}P_{efg}|e\rangle,P_{ab}P_{efg}|d\rangle,P_{cd}P_{ef}|g\rangle;P_{cd};P_{ab}) \\
&\quad -\frac{[b|P_{def}|g\rangle^2}{[a\,b]\,\langle d\,e\rangle\,\langle e\,f\rangle\,\langle f\,g\rangle\,[c|P_{abc}|g\rangle t_{abc}} \overline{H_3}(b,P_{abc}|g\rangle,P_{abc}|g\rangle;a,P_{abc}|d\rangle,c;P_{ab})\,,
\end{aligned}
\tag{4.2.33}
$$

where the spinors $|X_{D1}\rangle$ and $|X_{D4}\rangle$ are given by

$$
\begin{aligned}
|X_{D1}\rangle &= -\,|a\rangle[f|P_{ga}|c\rangle + |b\rangle\,[b\,f]\,\langle a\,c\rangle\,, \\
|X_{D4}\rangle &= |a\rangle[d|P_{ef}|g\rangle\,\langle c\,d\rangle + |c\rangle[b|P_{ef}|g\rangle\,\langle a\,b\rangle\,.
\end{aligned}
\tag{4.2.34}
$$

The D_E cut is lengthy; it is given by

$$D_E(a,b,c,d,e,f,g) = -\frac{1}{\langle c\,d\rangle \langle e\,f\rangle \langle f\,g\rangle s_{ab}}$$
$$\times \begin{pmatrix} [f|P_{eb}|d\rangle^2 & G_5(a,d;e,b,f,b,f;a,g,c,P_{fg}|e],P_{cd}|e];P_{efg};P_{ab}) \\ +[g|P_{eb}|d\rangle^2 & G_5(a,d;e,b,g,b,g;a,g,c,P_{fg}|e],P_{cd}|e];P_{efg};P_{ab}) \\ +[f\,a]^2\langle b\,d\rangle^2 & G_5(a,d;e,a,f,a,f;a,g,c,P_{fg}|e],P_{cd}|e];P_{efg};P_{ab}) \\ +[g\,a]^2\langle b\,d\rangle^2 & G_5(a,d;e,a,g,a,g;a,g,c,P_{fg}|e],P_{cd}|e];P_{efg};P_{ab}) \\ +2[f|P_{eb}|d\rangle[g|P_{eb}|d\rangle & G_5(a,d;e,b,f,b,g;a,g,c,P_{fg}|e],P_{cd}|e];P_{efg};P_{ab}) \\ +2[f|P_{eb}|d\rangle\,[f\,a]\,\langle b\,d\rangle & G_5(a,d;e,b,f,a,f;a,g,c,P_{fg}|e],P_{cd}|e];P_{efg};P_{ab}) \\ +2[f|P_{eb}|d\rangle\,[g\,a]\,\langle b\,d\rangle & G_5(a,d;e,b,f,a,g;a,g,c,P_{fg}|e],P_{cd}|e];P_{efg};P_{ab}) \\ +2[g|P_{eb}|d\rangle\,[f\,a]\,\langle b\,d\rangle & G_5(a,d;e,b,g,a,f;a,g,c,P_{fg}|e],P_{cd}|e];P_{efg};P_{ab}) \\ +2[g|P_{eb}|d\rangle\,[g\,a]\,\langle b\,d\rangle & G_5(a,d;e,b,g,a,g;a,g,c,P_{fg}|e],P_{cd}|e];P_{efg};P_{ab}) \\ +2\,[f\,a]\,\langle b\,d\rangle\,[g\,a]\,\langle b\,d\rangle & G_5(a,d;e,a,f,a,g;a,g,c,P_{fg}|e],P_{cd}|e];P_{efg};P_{ab}) \end{pmatrix}$$
$$-\frac{\langle e\,d\rangle}{\langle c\,d\rangle \langle e\,f\rangle \langle f\,g\rangle \langle d\,f\rangle s_{ab}}$$
$$\times \begin{pmatrix} [f|P_{eb}|d\rangle^2 & G_5(a,d;f,b,f,b,f;a,g,c,P_{fg}|e],P_{cd}|e];P_{fgX_{E3}};P_{ab}) \\ +[g|P_{eb}|d\rangle^2 & G_5(a,d;f,b,g,b,g;a,g,c,P_{fg}|e],P_{cd}|e];P_{fgX_{E3}};P_{ab}) \\ +[f\,a]^2\langle b\,d\rangle^2 & G_5(a,d;a,f,f,a,f;a,g,c,P_{fg}|e],P_{cd}|e];P_{fgX_{E3}};P_{ab}) \\ +[g\,a]^2\langle b\,d\rangle^2 & G_5(a,d;f,a,g,a,g;a,g,c,P_{fg}|e],P_{cd}|e];P_{fgX_{E3}};P_{ab}) \\ +2[f|P_{eb}|d\rangle[g|P_{eb}|d\rangle & G_5(a,d;f,b,f,b,g;a,g,c,P_{fg}|e],P_{cd}|e];P_{fgX_{E3}};P_{ab}) \\ +2[f|P_{eb}|d\rangle\,[f\,a]\,\langle b\,d\rangle & G_5(a,d;f,b,f,a,f;a,g,c,P_{fg}|e],P_{cd}|e];P_{fgX_{E3}};P_{ab}) \\ +2[f|P_{eb}|d\rangle\,[g\,a]\,\langle b\,d\rangle & G_5(a,d;f,b,f,a,g;a,g,c,P_{fg}|e],P_{cd}|e];P_{fgX_{E3}};P_{ab}) \\ +2[g|P_{eb}|d\rangle\,[f\,a]\,\langle b\,d\rangle & G_5(a,d;f,b,g,a,f;a,g,c,P_{fg}|e],P_{cd}|e];P_{fgX_{E3}};P_{ab}) \\ +2[g|P_{eb}|d\rangle\,[g\,a]\,\langle b\,d\rangle & G_5(a,d;f,b,g,a,g;a,g,c,P_{fg}|e],P_{cd}|e];P_{fgX_{E3}};P_{ab}) \\ +2\,[f\,a]\,\langle b\,d\rangle\,[g\,a]\,\langle b\,d\rangle & G_5(a,d;f,a,f,a,g;a,g,c,P_{fg}|e],P_{cd}|e];P_{fgX_{E3}};P_{ab}) \end{pmatrix}$$
$$+\frac{\langle d\,e\rangle^3}{\langle a\,b\rangle \langle e\,f\rangle \langle f\,g\rangle \langle d\,f\rangle [c|P_{de}|f\rangle} G_4(c,b;f,f,X_{E1b},X_{E1b};g,a,P_{ab}P_{cde}|f\rangle,P_{abc}P_{de}|f\rangle;P_{fgX_{E3}};P_{ab})$$
$$+\frac{[c|P_{cde}|b\rangle^2}{\langle f\,g\rangle \langle a\,b\rangle [c\,d]\,[d\,e]\,[c|P_{de}|f\rangle t_{cde}} H_3(b,P_{de}|c],P_{de}|c];a,g,P_{cd}|e];P_{ab})$$
$$-\frac{\langle d\,e\rangle^4\,[g\,a]^2}{\langle c\,d\rangle \langle d\,e\rangle \langle e\,f\rangle [a\,b]\,[g|P_{ab}|c\rangle t_{gab}} H_2(b,P_{ab}|g];a,P_{ab}P_{gab}|f\rangle;P_{ab})$$
$$-\frac{\langle d\,e\rangle^3\,[g|P_{def}|b\rangle^2}{\langle a\,b\rangle \langle e\,f\rangle [g|P_{def}|c\rangle[g|P_{ef}|d\rangle t_{abc}t_{def}} H_3(b,P_{def}|g],P_{def}|g];a,c,P_{abc}P_{de}|f\rangle;P_{ab})$$
$$-\frac{[f\,g]^3\,\langle b\,d\rangle^2}{\langle a\,b\rangle \langle c\,d\rangle [e\,f]\,[g|P_{ef}|d\rangle t_{efg}} H_3(b,d,d;a,c,P_{fg}|e];P_{ab})\,, \tag{4.2.35}$$

where the spinors $|X_{E1b}\rangle$ and $k_{X_{E3}}$ are defined as

$$\begin{aligned} |X_{E1b}\rangle &= [c\,g]\,\langle g\,f\rangle\,|b\rangle + [c\,a]\,\langle a\,b\rangle\,|f\rangle\,, \\ \not{k}_{X_{E3}} &= \frac{\langle e\,b\rangle}{\langle f\,d\rangle}|f\rangle\langle e|\,. \end{aligned} \tag{4.2.36}$$

Finally the cut D_F is given by

$$
\begin{aligned}
&D_F(a,b,c,d,e,f,g) = \\
&\frac{1}{\langle a\,b\rangle\,[c\,d]\,\langle e\,f\rangle\,\langle f\,g\rangle} G_4(c,a;f,f,X_{F1a},X_{F1a};b,g,P_{efg}|d],P_{ab}P_{cd}|e\rangle;P_{cd};P_{ab}) \\
&-\frac{[e\,c]}{\langle f\,g\rangle\,[d\,e]\,\langle a\,b\rangle} H_7(a,c,f,f,P_{abc}|e],X_{F1b},X_{F1b};b,g,c,P_{fg}|e],P_{fg}P_{de}|c\rangle,P_{abc}|d],P_{abc}P_{de}|f\rangle;P_{ab}) \\
&+\frac{1}{\langle f\,g\rangle\,[d\,e]\,\langle a\,b\rangle} \\
&H_7(a,f,f,P_{abc}|e],P_{ab}|e],X_{F1b},X_{F1b};b,g,c,P_{fg}P_{de}|c\rangle,P_{abc}P_{de}|f\rangle,P_{abc}|d],P_{fg}|e];P_{ab}) \\
&-\frac{1}{\langle f\,g\rangle\,\langle a\,b\rangle\,\langle c\,d\rangle\,\langle d\,e\rangle\,t_{cde}} \\
&\times G_5(P_{cde}|d],a;f,f,P_{ab}P_{cde}|d\rangle,X_{F1c},X_{F1c};b,g,P_{fg}P_{de}|c\rangle,P_{ab}P_{cd}|e\rangle,P_{ab}P_{cde}|f\rangle;P_{fg};P_{ab}) \\
&-\frac{\langle d\,f\rangle^4\,[g\,b]^2\,\langle b\,a\rangle}{\langle c\,d\rangle\,\langle d\,e\rangle\,\langle e\,f\rangle\,[g|P_{def}|c\rangle s_{ab} t_{gab}} \overline{H_2}(g,P_{ab}|a\rangle;P_{ab}|b\rangle,P_{cde}|f\rangle;P_{ab}) \\
&-\frac{\langle d\,f\rangle^4\,[a|P_{abc}|g\rangle^2}{\langle a\,b\rangle\,\langle d\,e\rangle\,\langle e\,f\rangle\,[g|P_{abc}|c\rangle[g|P_{abc}|d\rangle t_{abc} t_{def}} H_3(P_{abc}|g],P_{abc}|g],a;b,c,P_{abc}P_{de}|f\rangle;P_{ab}) \\
&-\frac{[g\,e]^4\,\langle d\,a\rangle^2}{\langle a\,b\rangle\,\langle c\,d\rangle\,[e\,f]\,[f\,g]\,[g|P_{efg}|d\rangle t_{efg}} H_3(d,d,a;b,c,P_{efg}|e];P_{ab})\,,
\end{aligned}
\tag{4.2.37}
$$

where the spinors $|X_{F1a}\rangle$, $|X_{F1b}\rangle$ and $|X_{F1c}\rangle$ are defined as

$$
\begin{aligned}
|X_{F1a}\rangle &= |f\rangle\,[c\,b]\,\langle b\,a\rangle + |a\rangle[c|P_{efg}|f\rangle\,, \\
|X_{F1b}\rangle &= |f\rangle\,[e\,d]\,\langle d\,a\rangle + \langle f\,a\rangle\,(|f\rangle\,[e\,f] + |g\rangle\,[e\,g])\,, \\
|X_{F1c}\rangle &= |a\rangle\,\langle f\,g\rangle\,[g|P_{cde}|d\rangle + |f\rangle\,\langle a\,b\rangle\,[b|P_{cde}|d\rangle\,.
\end{aligned}
\tag{4.2.38}
$$

With the general s-cuts solved, the specific cases found in the 7-point amplitude are given by

$$
\begin{aligned}
d_2^A &= D_D(2,3,4,5,6,7,1),\ d_3^A = -D_A(4,3,2,1,7,6,5), \\
d_4^A &= -D_E(5,4,3,2,1,7,6),\ d_7^A = D_B(7,1,2,3,4,5,6), \\
\\
d_2^B &= -D_C(2,1,7,6,5,4,3),\ d_4^B = D_E(4,5,3,2,1,7,6), \\
d_5^B &= -D_E(6,5,7,1,2,3,4),\ d_7^B = D_C(1,2,3,4,5,6,7), \\
\\
d_1^C &= -D_B(2,1,3,4,5,6,7),\ d_2^C = D_C(3,1,7,6,5,4,2), \\
d_3^C &= -D_C(3,5,6,7,1,2,4),\ d_4^C = D_B(4,5,3,2,1,7,6), \\
d_5^C &= D_F(5,6,7,1,2,3,4),\ d_7^C = -D_F(1,7,6,5,4,3,2),
\end{aligned}
\tag{4.2.39}
$$

With such large expressions for the double cuts, it is desirable to have a consistency check on the set of bubble coefficients beyond the checks performed on the canonical forms. Such a check is given by the IR behaviour of the loop amplitude; it is known that an $\mathcal{N}=1$ chiral loop amplitude should obey the IR constraint [84]

$$A_{IR}^{\mathcal{N}=1chiral} = \frac{c_\Gamma}{\epsilon} A^{tree}. \tag{4.2.40}$$

Since $\frac{1}{\epsilon}$ poles arise only from the scalar bubble integrals, this implies a consistency check upon the bubble coefficients,

$$\sum_i c_i + \sum_j d_j = A^{tree}.$$

This property has been verified numerically at a given point in momentum space for the above set of bubble coefficients, providing a useful confirmation of their accuracy.

4.3 Triangle Coefficients

Compared to the double cuts, there are relatively few unique cases which need be calculated for the triple cuts. As discussed in appendix (A), one can remove the need to compute the 1- and 2-mass triangles by exploiting the IR behaviour to absorb these triangles into the definition of the box integrals. As such we need only consider 3-mass triangles. The situation is even simpler however due to the fact that at seven point, all possible three-mass triangles contain only MHV corners. We can thus consider a single, general triple cut for the $\mathcal{N}=1$ chiral loop,

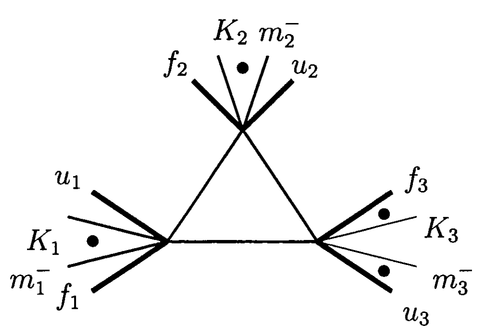


Figure 4.3: The three mass triangle appearing in the NMHV amplitudes

$$\sum_h A(-\ell_0^h, \cdots, m_1^-, \cdots, \ell_1^{-h}) \times A(-\ell_1^h, \cdots, m_2^-, \cdots, \ell_2^{-h}) \times A(-\ell_2^h, \cdots, m_3^-, \cdots, \ell_0^{-h}), \tag{4.3.1}$$

where $\ell_1 = \ell_0 - K_1$ etc. As before, the summation is over the $\mathcal{N}=1$ chiral multiplet, and thus h takes the value 0 for a complex scalar circulating, and ± 1 for the fermionic contributions. The effect of this summation is to give the Scalar contribution times a ρ-factor,

$$A(-\ell_0^s, \cdots, m_1^-, \cdots, \ell_1^s) \times A(-\ell_1^s, \cdots, m_2^-, \cdots, \ell_2^s) \times A(-\ell_2^s, \cdots, m_3^-, \cdots, \ell_0^s) \times \rho\,,$$

where

$$\begin{aligned}\rho &= \frac{(\langle m_1\,\ell_1\rangle\,\langle m_2\,\ell_2\rangle\,\langle m_3\,\ell_0\rangle - \langle m_1\,\ell_0\rangle\,\langle m_2\,\ell_1\rangle\,\langle m_3\,\ell_2\rangle)^2}{\langle m_1\,\ell_1\rangle\,\langle m_2\,\ell_2\rangle\,\langle m_3\,\ell_0\rangle\,\langle m_1\,\ell_0\rangle\,\langle m_2\,\ell_1\rangle\,\langle m_3\,\ell_2\rangle}\,,\\ &= \frac{\langle \ell_0\,X\rangle^2}{[\ell_1\,\ell_2]^2\,\langle m_1\,\ell_1\rangle\,\langle m_2\,\ell_2\rangle\,\langle m_3\,\ell_0\rangle\,\langle m_1\,\ell_0\rangle\,\langle m_2\,\ell_1\rangle\,\langle m_3\,\ell_2\rangle}\,,\end{aligned} \tag{4.3.2}$$

and

$$|X\rangle = |m_1\rangle\langle m_3|K_3K_2|m_2\rangle + |m_3\rangle\langle m_1|K_1K_2|m_2\rangle\,. \tag{4.3.3}$$

The cut is then

$$\begin{aligned}&\frac{\langle m_1\,\ell_0\rangle\,\langle m_1\,\ell_1\rangle}{\langle \ell_0\,f_1\rangle\,\langle f_1\cdots u_1\rangle\,\langle u_1\,\ell_1\rangle\,\langle \ell_1\,\ell_0\rangle} \times \frac{\langle m_2\,\ell_1\rangle\,\langle m_2\,\ell_2\rangle}{\langle \ell_1\,f_2\rangle\,\langle f_2\cdots u_2\rangle\,\langle u_2\,\ell_2\rangle\,\langle \ell_2\,\ell_1\rangle}\\ &\times \frac{\langle m_3\,\ell_2\rangle\,\langle m_3\,\ell_0\rangle}{\langle \ell_2\,f_3\rangle\,\langle f_3\cdots u_3\rangle\,\langle u_3\,\ell_0\rangle\,\langle \ell_0\,\ell_2\rangle} \times \frac{\langle \ell_0\,X\rangle^2}{[\ell_1\,\ell_2]^2}\,,\\ &= C_0 \times \frac{\langle m_1\,\ell_0\rangle\,\langle m_1\,\ell_1\rangle}{\langle \ell_0\,f_1\rangle\,\langle u_1\,\ell_1\rangle} \times \frac{\langle m_2\,\ell_1\rangle\,\langle m_2\,\ell_2\rangle}{\langle \ell_1\,f_2\rangle\,\langle f_2\,\ell_2\rangle} \times \frac{\langle m_3\,\ell_2\rangle\,\langle m_3\,\ell_0\rangle}{\langle \ell_2\,f_3\rangle\,\langle u_3\,\ell_0\rangle} \times \frac{\langle X\,\ell_0\rangle^2}{\langle \ell_0|\ell_1\ell_2|\ell_0\rangle}\,,\end{aligned} \tag{4.3.4}$$

where

$$C_0 = \frac{\langle u_1\,f_2\rangle\,\langle u_2\,f_3\rangle\,\langle u_3\,f_1\rangle}{\prod_i \langle i\,i+1\rangle\,K_2^2}\,. \tag{4.3.5}$$

This can be turned into a function ℓ_0 only

$$C_0 \times \frac{\prod_{y\in T_1}\langle \ell_0\,y\rangle}{\prod_{x\in S}\langle \ell_0\,x\rangle} \times \frac{1}{\langle \ell_0|K_1K_3|\ell_0\rangle}\,, \tag{4.3.6}$$

where

$$\begin{aligned}&S = \{|f_1\rangle, |u_3\rangle, K_3K_2|f_2\rangle, K_3K_2|u_1\rangle, K_1K_2|f_3\rangle, K_1K_2|u_2\rangle\},\\ &T_1 = \{|m_1\rangle, |m_3\rangle, K_3K_2|m_1\rangle, K_3K_2|m_2\rangle, K_1K_2|m_2\rangle, K_1K_2|m_3\rangle, |X\rangle, |X\rangle\},\end{aligned} \tag{4.3.7}$$

where we have used,

$$\frac{\langle \ell_1\,a\rangle}{\langle \ell_1\,b\rangle} = \frac{\langle \ell_0|\ell_2\ell_1|a\rangle}{\langle \ell_0|\ell_2\ell_1|b\rangle} = \frac{\langle \ell_0|K_3K_2|a\rangle}{\langle \ell_0|K_3K_2|b\rangle}, \qquad \frac{\langle \ell_2\,a\rangle}{\langle \ell_2\,b\rangle} = \frac{\langle \ell_0|K_1K_2|a\rangle}{\langle \ell_0|K_1K_2|b\rangle}. \tag{4.3.8}$$

This is precisely the canonical form $\mathcal{J}_n^0$ with $n=6$ as defined in [79]. So the three mass triangle coefficient is precisely

$$b_3^{3m}(K_1, K_2, K_3, m_1, m_2, m_3) = C_0 \times J_6^0(S; T_1; K_i)\,. \tag{4.3.9}$$

This general expression simplifies in many cases: if the m_i coincide with any of the u_i or f_i the J_6^0 function simplifies to a J_n^0 with $n < 3$.

The triangle coefficients in the NMHV amplitude can thus be simply evaluated from this expression,

$$
\begin{aligned}
b_1^A &= b_3^{3m}(K_{23}, K_{45}, K_{671}, 2, 4, 1),\ b_2^A = b_3^{3m}(K_{71}, K_{23}, K_{456}, 1, 2, 4),\\
\\
b_1^B &= b_3^{3m}(K_{23}, K_{45}, K_{671}, 2, 5, 1),\ b_2^B = b_3^{3m}(K_{71}, K_{23}, K_{456}, 1, 2, 5),\\
&\qquad b_3^B = b_3^{3m}(K_{56}, K_{71}, K_{234}, 5, 1, 2),\\
\\
b_1^C &= b_3^{3m}(K_{12}, K_{34}, K_{567}, 1, 3, 5),\ b_2^C = b_3^{3m}(K_{712}, K_{34}, K_{56}, 1, 3, 5),\\
b_3^C &= b_3^{3m}(K_{71}, K_{23}, K_{456}, 1, 3, 5),\ b_4^C = b_3^{3m}(K_{71}, K_{234}, K_{56}, 1, 3, 5),\\
&\qquad b_5^C = b_3^{3m}(K_{671}, K_{23}, K_{45}, 1, 3, 5),
\end{aligned}
\tag{4.3.10}
$$

4.4 Box Coefficients

Once again, although there are some 38 distinct box coefficients distributed among the 3 partial amplitudes, many of these are in fact identical up to a relabeling or flip and we can thus solve for a relatively small basis of general quadruple cuts instead of working term-by-term. In addition, since all 3-mass and 2-mass boxes contain no higher than 5-point trees in a 7-point amplitude, these quadruple cuts can be solved for the general case; the only case which cannot be solved in general is the 1-mass box with an 6-point, NMHV corner; this will be solved case-by-case for the four possible helicity configurations.

4.4.1 3-mass

The 3-mass boxes appearing in the 7-point amplitude can be described solving the general case with positive massless leg and MHV massive corners,

Inserting the expressions for the trees we get the cut for circulating particle helicity h to be

$$
\begin{aligned}
&C^{3m}(m_1, m_2, m_3, d, K_2, K_3, K_4, h) =\\
&\quad \frac{[d\,l_1]^2\,[d\,l_4]^2}{[d\,l_1]\,[l_1\,l_4]\,[l_4\,d]}\,\frac{\langle m_1\,l_1\rangle^2\,\langle m_1\,l_2\rangle^2}{\langle u_2\,l_2\rangle\,\langle l_2\,l_1\rangle\,\langle l_1\,f_2\rangle}\,\frac{\langle m_2\,l_2\rangle^2\,\langle m_2\,l_3\rangle^2}{\langle u_3\,l_3\rangle\,\langle l_3\,l_2\rangle\,\langle l_2\,f_3\rangle}\,\frac{\langle m_3\,l_3\rangle^2\,\langle m_3\,l_4\rangle^2}{\langle u_4\,l_4\rangle\,\langle l_4\,l_3\rangle\,\langle l_3\,f_4\rangle}\\
&\quad \times\frac{\langle d\,f_2\rangle\,\langle u_2\,f_3\rangle\,\langle u_3\,f_4\rangle\,\langle u_4\,d\rangle}{\langle 1\,2\rangle\,\langle 2\,3\rangle\cdots\langle n\,1\rangle}\times\left(\frac{[d\,l_1]\,\langle m_1\,l_1\rangle\,\langle m_2\,l_2\rangle\,\langle m_3\,l_3\rangle}{[d\,l_4]\,\langle m_1\,l_2\rangle\,\langle m_2\,l_3\rangle\,\langle m_3\,l_4\rangle}\right)^h,
\end{aligned}
\tag{4.4.1}
$$

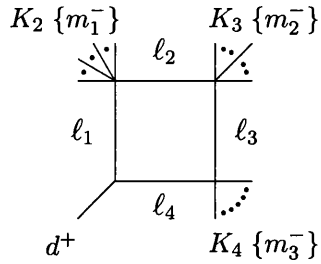


Figure 4.4: 3-mass box with MHV corner

where as before the helicity h takes the value 0 for a scalar circulating in the loop, and ± 1 for the fermionic contributions. The solution for this class of cut is given in Appendix C, up to a simple conjugation. We now apply it to first solve for the supersymmetric sum,

$$\rho^{\mathrm{SUSY}} = \frac{([d|l_1|m_1\rangle \langle m_2\, l_2\rangle \langle m_3\, l_3\rangle + [d|l_4|m_3\rangle \langle m_1\, l_2\rangle \langle m_2\, l_3\rangle)^2}{[d\, l_1]\, [d\, l_4]\, \langle m_1\, l_1\rangle \langle m_1\, l_2\rangle \langle m_2\, l_2\rangle \langle m_2\, l_3\rangle \langle m_3\, l_3\rangle \langle m_3\, l_4\rangle}\,. \tag{4.4.2}$$

The numerator can be simplified using the symmetries arising from momentum conservation,

$$\begin{aligned}
&([d|l_1|m_1\rangle \langle m_2\, l_2\rangle \langle m_3\, l_3\rangle + [d|l_4|m_3\rangle \langle m_1\, l_2\rangle \langle m_2\, l_3\rangle)^2 = \\
&\qquad \frac{\begin{array}{l}([d|(K_2+K_3)K_3K_4|d\rangle \langle d\, m_1\rangle \langle m_2|K_3K_4|d\rangle \langle m_3|K_3K_2|d\rangle \\ -[d|(K_3+K_4)K_3K_4|d\rangle \langle d\, m_3\rangle \langle m_1|K_3K_4|d\rangle \langle m_2|K_3K_2|d\rangle)^2\end{array}}{\langle d|K_2K_3|d\rangle^6}\,, \\
&\quad = \frac{[d|K_2K_3K_4|d\rangle^2}{\langle d|K_2K_3|d\rangle^6}(\langle d\, m_1\rangle \langle m_2|K_3K_4|d\rangle \langle m_3|K_3K_2|d\rangle - \langle d\, m_3\rangle \langle m_1|K_3K_4|d\rangle \langle m_2|K_3K_2|d\rangle)^2\,, \\
&\quad = \frac{[d|K_2K_3K_4|d\rangle^2}{\langle d|K_2K_3|d\rangle^6}(\langle m_1\, m_2\rangle \langle d|K_3K_4|d\rangle \langle m_3|K_3K_2|d\rangle + \langle d\, m_2\rangle \langle m_1|K_3K_4|d\rangle \langle m_3|K_3K_2|d\rangle \\
&\qquad - \langle m_3\, m_2\rangle \langle d|K_3K_2|d\rangle \langle m_1|K_3K_4|d\rangle - \langle d\, m_2\rangle \langle m_1|K_3K_4|d\rangle \langle m_3|K_3K_2|d\rangle)^2\,, \\
&\quad = \frac{[d|K_2K_3K_4|d\rangle^2}{\langle d|K_2K_3|d\rangle^4}(\langle m_1\, m_2\rangle \langle d|K_2K_3|m_3\rangle + \langle m_3\, m_2\rangle \langle d|K_4K_3|m_1\rangle)^2\,.
\end{aligned} \tag{4.4.3}$$

Using this, the full cut can be solved yielding the expression,

$$\begin{aligned}
&C^{3m}(m_1, m_2, m_3, d, K_1, K_2, K_3) = \\
&\qquad -\frac{(\langle m_1\, m_2\rangle \langle d|K_2K_3|m_3\rangle + \langle m_3\, m_2\rangle \langle d|K_4K_3|m_1\rangle)^2 \langle u_2\, f_3\rangle \langle u_3\, f_4\rangle \langle m_3\, d\rangle \langle m_1\, d\rangle}{2\, \langle 1\, 2\rangle \langle 2\, 3\rangle \cdots \langle n\, 1\rangle} \\
&\quad \times \frac{\langle m_1|K_3K_4|d\rangle \langle m_3|K_3K_2|d\rangle \langle m_2|K_3K_2|d\rangle \langle m_2|K_3K_4|d\rangle [d|K_2K_3K_4|d\rangle}{\langle d|K_2K_3|d\rangle^2 \langle d|K_4K_3|u_2\rangle \langle d|K_4K_3|f_3\rangle \langle d|K_2K_3|u_3\rangle \langle d|K_2K_3|f_4\rangle K_3^2}\,.
\end{aligned} \tag{4.4.4}$$

4.4.2 2-mass

The two 2-mass quadruple cuts required consist of the 2-mass hard box with MHV corners, and the 2-mass easy box with opposing MHV and $\overline{\text{MHV}}$ massive corners,

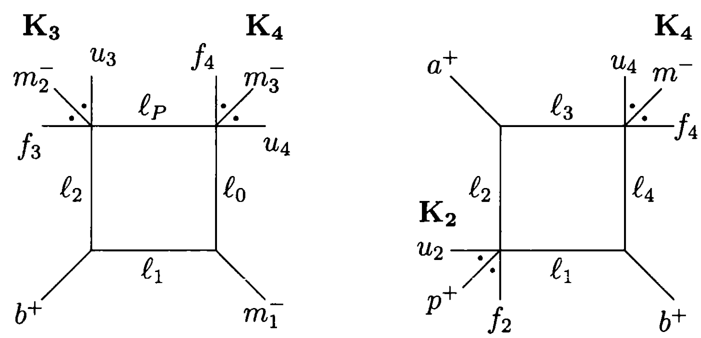


Figure 4.5: 2-mass hard (left) and easy (right) boxes with MHV corners

The general 2-mass easy configuration yields a cut of the form

$$\begin{aligned}C^{2me}(a,b,m,p,K_2,K_4) =& \frac{[a\,l_2]^2\,[a\,l_3]^2}{[a\,l_3]\,[l_3\,l_2]\,[l_2\,a]}\frac{\langle m\,l_3\rangle^2\,\langle m\,l_4\rangle^2}{\langle u_4\,l_4\rangle\,\langle l_4\,l_3\rangle\,\langle l_3\,f_4\rangle}\frac{[b\,l_4]^2\,[b\,l_1]^2}{[b\,l_1]\,[l_1\,l_4]\,[l_4\,b]}\frac{[p\,l_1]^2\,[p\,l_2]^2}{[u_2\,l_2]\,[l_2\,l_1]\,[l_1\,f_2]}\\ &\times\frac{1}{\prod_{i=f_4}^{u_4-1}\langle i\,i+1\rangle\prod_{i=f_2}^{u_2-1}[i\,i+1]}\left(\frac{[a\,l_2]\,\langle m\,l_4\rangle\,[b\,l_4]\,[p\,l_1]}{[a\,l_3]\,\langle m\,l_3\rangle\,[b\,l_1]\,[p\,l_2]}\right)^h.\end{aligned} \tag{4.4.5}$$

This box can be evaluated using the solution derived in Appendix C, with the provision that the loop spinors adjacent to the negative massless corner be conjugated to account for the fact that we require the case with two positive massless corners. We can apply this to simplify the numerator of the supersymmetric contribution,

$$\begin{aligned}\rho^{\text{SUSY}} =&([b|l_4|m\rangle\,[a\,l_2]\,[p\,l_1] - [a|l_3|m\rangle\,[p\,l_2]\,[b\,l_1])^2\,,\\ =&\frac{(-[b|K_4|a\rangle\,\langle b\,m\rangle\,[a|K_2|b\rangle[p|K_2|a\rangle + [a|K_4|b\rangle\,\langle a\,m\rangle\,[p|K_2|b\rangle[b|K_2|a\rangle)^2}{\langle a\,b\rangle^6}\,,\\ =&\frac{[b|K_2|a\rangle^2[a|K_2|b\rangle^2}{\langle a\,b\rangle^6}(\langle b\,m\rangle\,[p|K_2|a\rangle - \langle a\,m\rangle\,[p|K_2|b\rangle)^2\,,\\ =&\frac{[b|K_2|a\rangle^2[a|K_2|b\rangle^2[p|K_2|m\rangle^2}{\langle a\,b\rangle^4}\,.\end{aligned} \tag{4.4.6}$$

The full cut is thus

$$\begin{aligned}
C^{2me} =&\frac{\langle m\,l_3\rangle\,\langle m\,l_4\rangle\,[p\,l_1]\,[p\,l_2]}{[l_1|l_4 l_3|l_2]\,[l_2\,l_1]\,\langle u_4\,l_4\rangle\,\langle f_4\,l_3\rangle\,[u_2\,l_2]\,[f_2\,l_1]}\times\frac{\rho^{\mathrm{SUSY}}}{\prod_{i=f_4}^{u_4-1}\langle i\,i+1\rangle\prod_{i=f_2}^{u_2-1}[i\,i+1]},\\
=&\frac{\langle m\,a\rangle\langle m\,b\rangle\,[p|K_2|a\rangle[p|K_2|b\rangle[b|K_2|a\rangle^2[a|K_2|b\rangle^2[p|K_2|m\rangle^2}{\langle a|K_2K_4|a\rangle\langle b|K_4K_2|b\rangle\,\langle u_4\,b\rangle\,\langle f_4\,a\rangle\,[u_2|K_2|b\rangle[f_2|K_2|a\rangle}\times\frac{1}{\prod_{i=f_4}^{u_4-1}\langle i\,i+1\rangle\prod_{i=f_2}^{u_2-1}[i\,i+1]},\\
=&\frac{[p|K_2|m\rangle^2[b|K_2|a\rangle[a|K_2|b\rangle[p|K_2|a\rangle[p|K_2|b\rangle\,\langle m\,a\rangle\,\langle m\,b\rangle}{2K_2^2\,\langle a\,f_4\rangle\,\langle u_4\,b\rangle\,\langle a\,b\rangle^2\,[f_2|K_2|a\rangle[u_2|K_2|b\rangle\prod_{i=f_4}^{u_4-1}\langle i\,i+1\rangle\prod_{i=f_2}^{u_2-1}[i\,i+1]}.
\end{aligned}\tag{4.4.7}$$

The only 2-mass hard boxes which can appear at 7-point are those of the form shown in figure 4.4.2, plus those given by flipping the positive and negative massless legs. The general form consists purely of MHV and $\overline{\text{MHV}}$ corners, and is given by the expression

$$\begin{aligned}
&C^{2mh}(m_2,m_3,K_3,K_4,m_1,b,h)=\\
&\quad\frac{\langle m_1\,l_0\rangle^2\,\langle m_1\,l_1\rangle^2}{\langle m_1\,l_1\rangle\,\langle l_1\,l_0\rangle\,\langle l_0\,m_1\rangle}\frac{[b\,l_1]^2\,[b\,l_2]^2}{[b\,l_2]\,[l_2\,l_1]\,[l_1\,b]}\frac{\langle m_2\,l_P\rangle^2\,\langle m_2\,l_2\rangle^2}{\langle u_3\,l_P\rangle\,\langle l_P\,l_2\rangle\,\langle l_2\,f_3\rangle}\frac{\langle m_3\,l_P\rangle^2\,\langle m_3\,l_0\rangle^2}{\langle u_4\,l_0\rangle\,\langle l_0\,l_P\rangle\,\langle l_P\,f_4\rangle}\\
&\quad\times\frac{\langle u_3\,f_4\rangle}{\prod_{i=f_3}^{u_3-1}\langle i\,i+1\rangle\prod_{i=f_4}^{u_4-1}\langle i\,i+1\rangle}\left(\frac{\langle m_1\,l_0\rangle\,[b\,l_2]\,\langle m_2\,l_2\rangle\,\langle m_3\,l_P\rangle}{\langle m_1\,l_1\rangle\,[b\,l_1]\,\langle m_2\,l_P\rangle\,\langle m_3\,l_0\rangle}\right)^h.
\end{aligned}\tag{4.4.8}$$

The supersymmetric numerator is given by

$$\begin{aligned}
\rho^{\mathrm{SUSY}} =&([b\,l_2]\,\langle l_2\,m_2\rangle\,\langle l_P\,m_3\rangle\,\langle l_0\,m_1\rangle-[b\,l_1]\,\langle l_1\,m_1\rangle\,\langle l_0\,m_3\rangle\,\langle l_P\,m_2\rangle)^2,\\
=&\frac{1}{[m_1|P|b\rangle^6}(P^4\,[b\,m_1]\,\langle b\,m_2\rangle\,[m_1|P|m_3\rangle\,\langle b\,m_1\rangle+P^2\,[b\,m_1]\,\langle b\,m_1\rangle\,\langle b|PK_4|m_3\rangle[m_1|P|m_2\rangle)^2.
\end{aligned}\tag{4.4.9}$$

Applying the Schouten identity allows us to obtain

$$\begin{aligned}
\rho^{\mathrm{SUSY}} =&\frac{P^4\,[b\,m_1]^2}{[m_1|P|b\rangle^6}(P^2\,\langle b\,m_1\rangle\,([m_1|P|b\rangle\,\langle m_2\,m_3\rangle+[m_1|P|m_2\rangle\,\langle m_3\,b\rangle)\\
&+[m_1|P|m_2\rangle(P^2\,\langle b\,m_3\rangle\,\langle b\,m_1\rangle+\langle b|K_4P|b\rangle\,\langle m_1\,m_3\rangle))^2,\\
=&\frac{P^4\,[b\,m_1]^2}{[m_1|P|b\rangle^6}(P^2\langle b|m_1P|b\rangle\,\langle m_2\,m_3\rangle+\langle b|PK_4|b\rangle\langle m_3|m_1P|m_2\rangle)^2.
\end{aligned}\tag{4.4.10}$$

The full box coefficient is thus given by

$$C^{2mh}(m_2,m_3,K_3,K_4,m_1,b) =$$

$$\frac{P^2\,[m_1\,b]^2\,\langle m_3|P|m_1]\langle m_2|P|m_1]\,\langle m_2\,b\rangle\,\langle m_3|K_4P|b\rangle}{2K_4^2[m_1|K_3P|m_1]\langle b|PK_4|b\rangle\langle f_4|P|m_1]\langle u_3|P|m_1]\langle b|PK_4|u_4\rangle\,\langle f_3\,b\rangle\,[m_1|P|b\rangle^2} \times \frac{(P^2\langle b|m_1P|b\rangle\,\langle m_2\,m_3\rangle + \langle b|PK_4|b\rangle\langle m_3|m_1P|m_2\rangle)^2}{\prod_{i=f_3}^{u_3-1}\langle i\,i+1\rangle\prod_{i=f_4}^{u_4-1}\langle i\,i+1\rangle}\,. \tag{4.4.11}$$

4.4.3 1-mass

There are two distinct cases to consider for the 1-mass boxes possible in the 7-point amplitude. The first is the case with two negative, and one positive massless leg. In this case, the massive corner depends upon the 6-point MHV tree and can thus be solved for the general case. The cut has the structure

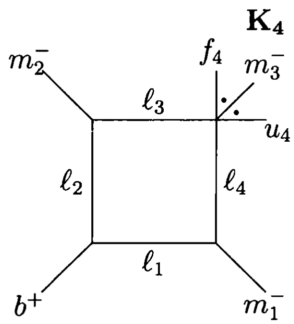


Figure 4.6: General 1-mass box with MHV corner

This cut has the loop momentum solution given in Appendix C. The cut is given by

$$C_0^{1m}(m_1,b,m_2,m_3,K_4,h) =$$

$$\frac{\langle m_1\,l_4\rangle^2\,\langle m_1\,l_1\rangle^2}{\langle l_4\,m_1\rangle\,\langle m_1\,l_1\rangle\,\langle l_1\,l_4\rangle}\,\frac{[b\,l_1]^2\,[b\,l_2]^2}{[b\,l_2]\,[l_2\,l_1]\,[l_1\,b]}\,\frac{\langle m_2\,l_2\rangle^2\,\langle m_2\,l_3\rangle^2}{\langle m_2\,l_3\rangle\,\langle l_3\,l_2\rangle\,\langle l_2\,m_2\rangle}\,\frac{\langle m_3\,l_3\rangle^2\,\langle m_3\,l_4\rangle^2}{\langle u_4\,l_4\rangle\,\langle l_4\,l_3\rangle\,\langle l_3\,f_4\rangle} \times\frac{1}{\prod_{i=f_4}^{u_4-1}\langle i\,i+1\rangle}\left(\frac{[b\,l_1]\,\langle m_1\,l_1\rangle\,\langle m_3\,l_4\rangle\,\langle m_2\,l_3\rangle}{[b\,l_2]\,\langle m_1\,l_4\rangle\,\langle m_3\,l_3\rangle\,\langle m_2\,l_2\rangle}\right)^h\,. \tag{4.4.12}$$

The supersymmetric numerator has the simplification

$$\begin{aligned}\rho^{\text{SUSY}} =& ([b|l_1|m_3\rangle\,\langle m_3\,l_4\rangle\,\langle m_2\,l_3\rangle - [b|l_2|m_2\rangle\,\langle m_1\,l_4\rangle\,\langle m_3\,l_3\rangle)^2\,,\\ =&\frac{s_{bm_1}^2 s_{bm_2}^2}{[m_1\,m_2]^6}([b\,m_1]\,[m_2|K_4|m_3\rangle - [b\,m_2]\,[m_3|K_4|m_3\rangle)^2\,,\\ =&\frac{s_{bm_1}^2 s_{bm_2}^2\,[b|K_4|m_3\rangle^2}{[m_1\,m_2]^4}\,.\end{aligned} \tag{4.4.13}$$

The solution to the cut is thus given by

$$
\begin{aligned}
C_0^{1m}(m_1,b,m_2,m_3,K_4) =& \frac{1}{\langle l_1\, l_4\rangle\,[l_2\, l_1]\,\langle l_3\, l_2\rangle\,\langle l_4\, l_3\rangle}\,\frac{\langle m_3\, l_3\rangle\,\langle m_3\, l_4\rangle}{\langle u_4\, l_4\rangle\,\langle l_3\, f_4\rangle}\,\frac{\rho^{\mathrm{SUSY}}}{\prod_{i=f_4}^{u_4-1}\langle i\, i+1\rangle}\,, \\
=& \frac{[m_1|K_4|m_3\rangle[m_2|K_4|m_3\rangle[b|K_4|m_3\rangle^2 s_{bm_1}s_{bm_2}}{2K_4^2\,[m_1\, m_2]^2\,[m_2|K_4|u_4\rangle[m_1|K_4|f_4\rangle\prod_{i=f_4}^{u_4-1}\langle i\, i+1\rangle}\,.
\end{aligned}
\tag{4.4.14}
$$

The other possible 1-mass box case, that of a quadruple cut with two positive and one negative massless leg, contains a massive corner with an NMHV 6-point tree amplitude. There is thus no obvious way to solve such a cut in general, and instead the four possible cases corresponding to the possible helicity configuration on the 6-point tree must be considered individually.

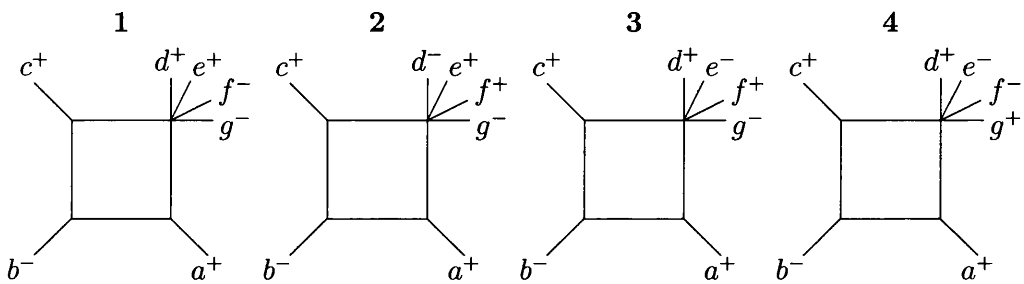


Figure 4.7: Specific NMHV 1-mass box coefficients

The derivation of the cuts for these cases proceeds identically to that for the MHV 1-mass box, and as such only the final results for each case are presented here.

$$
C_1^{1m} = \frac{\langle b|P_{abc}P_{def}|f\rangle^2\langle c|P_{abc}P_{def}|f\rangle s_{ab}s_{bc}}{2t_{def}t_{abc}\,\langle a\, c\rangle^2\,\langle d\, e\rangle\,\langle e\, f\rangle\,[g|P_{ab}|c\rangle[g|P_{def}|d\rangle} - \frac{[e|P_{efg}|b\rangle^2[e|P_{efg}|c\rangle s_{ab}s_{bc}}{2\,\langle a\, c\rangle^2\,[e\, f]\,[f\, g]\,\langle c\, d\rangle\,[g|P_{efg}|d\rangle t_{efg}}\,, \tag{4.4.15}
$$

$$
\begin{aligned}
C_2^{1m} =& \frac{s_{ab}s_{bc}\,\langle b\, g\rangle^2\,\langle c\, g\rangle\,[e\, f]^3}{2t_{def}\,\langle a\, c\rangle^2\,[d\, e]\,[d|P_{def}|g\rangle[f|P_{def}|c\rangle} \\
&+\frac{s_{ab}s_{bc}([f\, g]\,\langle g\, d\rangle\,\langle b\, c\rangle - [f|P_{abc}|c\rangle\,\langle d\, b\rangle)^2([f\, g]\,\langle g\, d\rangle\,\langle a\, c\rangle - [f|P_{abc}|c\rangle\,\langle d\, a\rangle)[f|P_{abc}|c\rangle}{2\,\langle a\, c\rangle^2\,[f\, g]\,[g|P_{abc}|c\rangle\,\langle d\, e\rangle\,[f|P_{de}|c\rangle\langle c|P_{abc}P_{fg}|e\rangle(s_{de}\,\langle c\, a\rangle + \langle c|P_{de}P_{bc}|a\rangle)} \\
&-\frac{s_{ab}s_{bc}\langle b|P_{abc}P_{efg}|g\rangle^2\langle a|P_{abc}P_{efg}|g\rangle\langle c|P_{abc}P_{efg}|g\rangle}{2t_{efg}t_{abc}\,\langle a\, c\rangle^2\,[d|P_{efg}|g\rangle[d|P_{abc}|a\rangle\,\langle e\, f\rangle\,\langle f\, g\rangle\,\langle c|P_{abc}P_{efg}|e\rangle}\,,
\end{aligned}
\tag{4.4.16}
$$

$$
\begin{aligned}
C_3^{1m} =& \frac{s_{ab}s_{bc}\langle b|P_{abc}P_{def}|e\rangle^2\langle c|P_{abc}P_{def}|e\rangle\langle a|P_{abc}P_{def}|e\rangle}{2t_{def}t_{abc}\,\langle c\, a\rangle^2\,\langle d\, e\rangle\,\langle e\, f\rangle\,[g|P_{abc}|c\rangle[g|P_{def}|d\rangle\langle a|P_{abc}P_{def}|f\rangle} \\
&-\frac{s_{ab}s_{bc}[f|P_{efg}|b\rangle^2[f|P_{efg}|a\rangle[f|P_{efg}|c\rangle}{2t_{efg}\,\langle a\, c\rangle^2\,\langle c\, d\rangle\,[e\, f]\,[f\, g]\,[e|P_{efg}|a\rangle[g|P_{efg}|d\rangle} \\
&-\frac{s_{ab}s_{bc}([d|P_{abc}|a\rangle\,\langle g\, b\rangle + [d\, e]\,\langle e\, g\rangle\,\langle a\, b\rangle)^2([d|P_{abc}|a\rangle\,\langle g\, c\rangle + [d\, e]\,\langle e\, g\rangle\,\langle a\, c\rangle)}{2\,\langle c\, a\rangle^2\,\langle f\, g\rangle\,[d\, e]\,[e|P_{efg}|a\rangle\langle a|P_{abc}P_{def}|f\rangle(s_{de}\,\langle c\, a\rangle + \langle c|P_{de}P_{abc}|a\rangle)}\,,
\end{aligned}
\tag{4.4.17}
$$

$$
\begin{aligned}
C_4^{1m} = &- \frac{s_{ab}s_{bc}\left\langle f\,a\right\rangle (\left\langle f\,a\right\rangle [d|P_{abc}|b\rangle - \left\langle a\,b\right\rangle [d\,g]\left\langle g\,f\right\rangle)^2(\left\langle f\,a\right\rangle [d|P_{abc}|c\rangle - \left\langle a\,c\right\rangle [d\,g]\left\langle g\,f\right\rangle)}{2\left\langle c\,a\right\rangle^2 \left\langle f\,g\right\rangle \left\langle g\,a\right\rangle [d\,e]\,\langle a|P_{abc}P_{def}|f\rangle [e|P_{efg}|a\rangle (s_{de}\left\langle c\,a\right\rangle + \langle c|P_{de}P_{abc}|a\rangle)} \\
&+ \frac{s_{ab}s_{bc}[g|P_{abc}|b\rangle^2 [g|P_{abc}|a\rangle \left\langle e\,f\right\rangle^3}{2\left\langle a\,c\right\rangle^2 t_{def}t_{abc}\left\langle d\,e\right\rangle [g|P_{def}|d\rangle \langle a|P_{abc}P_{def}|f\rangle} \\
&- \frac{s_{ab}s_{bc}[g|P_{efg}|b\rangle^2 [g|P_{efg}|a\rangle [g|P_{efg}|c\rangle}{2t_{efg}\left\langle a\,c\right\rangle^2 \left\langle c\,d\right\rangle [e\,f]\,[f\,g]\,[e|P_{efg}|a\rangle [g|P_{efg}|d\rangle}\,.
\end{aligned} \tag{4.4.18}
$$

Chapter 5

The 6-gluon NMHV complex scalar loop amplitude

5.1 Motivation and General Structure

In Chapter 4 the canonical basis method was applied to a non-trivial, previously unknown loop amplitude, the $\mathcal{N} = 1$ 7-gluon NMHV loop. According to equation (2.2.6) however, in order to calculate a gluon or adjoint fermion loop one needs both the $\mathcal{N} = 1$ chiral component, and the contribution from the complex scalar loop (as well as the relatively simple $\mathcal{N} = 4$ SYM multiplet loop for the gluon loop case). It is thus useful to demonstrate the applicability of the canonical basis method to both non-trivial parts of the supersymmetric decomposition by calculating the cut-constructible parts of a complex scalar loop of significant difficulty.

The chosen amplitude to be calculated is the 6-gluon NMHV scalar loop. This amplitude has been previously calculated [82] using the semi-numerical approach of Ellis, Giele and Zanderighi [43] (a review of the various 6-point partial amplitudes is also given in [83]), which provides a useful test of the canonical basis result in addition to the tests applied to the 7-point. It is also the simplest amplitude which contains NMHV trees in its double cuts and thus will give rise to the most difficult class of canonical forms, the quadratic G-functions. It also has the benefit that a great deal of the structure can be reapplied to the 7-point scalar loop, as will be examined in section 5.5.

It is important to note that the integral basis for a complex scalar loop is different to that for a SUSY multiplet loop. Specifically, since there is no supersymmetric cancellation present, we instead have the full integral basis including rational terms,

$$A_6 = \sum_{i\in\mathcal{C}} c_i \mathcal{F}_i^4 + \sum_{j\in\mathcal{D}} d_j I_j^{3m} + \sum_{k\in\mathcal{E}} e_k I_k^2 + \mathcal{R}\,. \tag{5.1.1}$$

The rational terms cannot be obtained via the canonical basis method in its current form, since it is an implementation of 4-dimensional Unitarity. As such this thesis will only consider the calculation of the cut-constructible parts, as the rational terms are of little interest as an illustration of the canonical basis method (although obviously they must be calculated in order to have a complete expression of the integral for phenomenological purposes; a discussion of possible methods for doing so, and Unitarity implementations which have tackled this problem is contained in Chapters 1 and 2).

The 6-point scalar loop consists of 3 primitive amplitudes corresponding to the possible helicity configurations,

$$
\begin{aligned}
A :&A_6(1^-,2^-,3^-,4^+,5^+,6^+)\,,\\
B :&A_6(1^-,2^-,3^+,4^-,5^+,6^+)\,,\\
C :&A_6(1^-,2^+,3^-,4^+,5^-,6^+)\,.
\end{aligned}
\tag{5.1.2}
$$

Another advantage of the 6-point is that it contains a great deal more inherent symmetry in all three amplitudes than the 7-point, which provides additional consistency checks on the results. Specifically, the A amplitude is symmetric under both a flip and conjugation of the internal particles, and under a cycling by three and parity conjugation,

$$
\begin{aligned}
A_6^A(1^-,2^-,3^-,4^+,5^+,6^+) =&\overline{A_6^A(6^+,5^+,4^+,3^-,2^-,1^-)}\,,\\
=&\overline{A_6(4^+,5^+,6^+,1^-,2^-,3^-)}\,.
\end{aligned}
\tag{5.1.3}
$$

The B amplitude, meanwhile, is symmetric under a flip and conjugation,

$$
A_6^B(1^-,2^-,3^+,4^-,5^+,6^+) = \overline{A_6^A(6^+,5^+,4^-,3^+,2^-,1^-)}\,. \tag{5.1.4}
$$

The C amplitude has a great deal of symmetry; under a simple cycling by two or four, and under a flip and conjugation,

$$
\begin{aligned}
A_6^C(1^-,2^+,3^-,4^+,5^-,6^+) =&A_6^C(3^-,4^+,5^-,6^+,1^-,2^+) = A_6^C(5^-,6^+,1^-,2^+,3^-,4^+)\,,\\
=&\overline{A_6^C(6^+,5^-,4^+,3^-,2^+,1^-)}\,.
\end{aligned}
\tag{5.1.5}
$$

These symmetries also come into play in determining the basis of integrals with non-vanishing coefficients. Consider first the A amplitude,

$$
\begin{aligned}
A_6^A(1^-,2^-,3^-,4^+,5^+,6^+) =&c_2^A I^2(t_{234}) + c_3^A I^2(t_{345})\,,\\
&+ d_3^A I^2(s_{34}) + d_6^A I^2(s_{61}) + \mathcal{R}\,.
\end{aligned}
\tag{5.1.6}
$$

Note that although we might expect the cuts t_{561} and t_{612} to be present, these integrals are in fact equivalent to the t_{234} and t_{345} integrals respectively by momentum conservation, and thus are not independent momentum channels.

The B amplitude has the integral structure

$$
\begin{aligned}
A_6^B(1^-,2^-,3^+,4^-,5^+,6^+) =\;& a_1^B\mathcal{F}^{2mh}_{1\{23\}\{45\}6} + a_2^B\mathcal{F}^{2mh}_{3\{45\}\{61\}2} + a_3^B\mathcal{F}^{2mh}_{5\{61\}\{23\}4}\\
&+ a_4^B\mathcal{F}^{1m}_{234\{561\}} + a_4^B\mathcal{F}^{1m}_{345\{612\}}\\
&+ b_1^B I^{3m}_{\{23\}\{45\}\{61\}}\\
&+ c_1^B I^2(t_{123}) + c_2^B I^2(t_{234}) + c_3^B I^2(t_{345})\\
&+ d_2^B I^2(s_{23}) + d_3^B I^2(s_{34}) + d_4^B I^2(s_{45}) + d_5^B I^2(s_{61}) + \mathcal{R}\,.
\end{aligned}
\tag{5.1.7}
$$

Note that the symmetry reduces the number of independent 3-mass triangle coefficients we must calculate - as with the t-cuts, seemingly independent valid cuts turn out to be the same momentum channel.

The C amplitude is

$$
\begin{aligned}
A_6^C(1^-,2^+,3^-,4^+,5^-,6^+) = {} & a_1^C \mathcal{F}^{2mh}_{\{12\}\{34\}56} + a_2^C \mathcal{F}^{2mh}_{\{23\}\{45\}61} + a_3^C \mathcal{F}^{2mh}_{\{34\}\{56\}12} + a_4^C \mathcal{F}^{2mh}_{\{45\}\{61\}23} \\
& + a_5^C \mathcal{F}^{2mh}_{\{56\}\{12\}34} + a_6^C \mathcal{F}^{2mh}_{\{61\}\{23\}45} + a_7^C \mathcal{F}^{1m}_{123\{456\}} + a_8^C \mathcal{F}^{1m}_{234\{561\}} \\
& + a_9^C \mathcal{F}^{1m}_{345\{612\}} + a_{10}^C \mathcal{F}^{1m}_{456\{123\}} + a_{11}^C \mathcal{F}^{1m}_{561\{234\}} + a_{12}^C \mathcal{F}^{1m}_{612\{345\}} \\
& + b_1^C I^{3m}_{\{12\}\{34\}\{56\}} + b_2^C I^{3m}_{\{23\}\{45\}\{61\}} \\
& + c_1^C I^2(t_{123}) + c_2^C I^2(t_{234}) + c_3^C I^2(t_{345}) \\
& + d_1^C I^2(s_{12}) + d_2^C I^2(s_{23}) + d_3^C I^2(s_{34}) + d_4^C I^2(s_{45}) + d_5^C I^2(s_{56}) + d_6^C I^2(s_{61}) \\
& + \mathcal{R}\,.
\end{aligned}
\tag{5.1.8}
$$

Again symmetry halves the number of t-cuts, and reduces the number of 3-mass triangles by two thirds. The cyclic helicity symmetry of amplitude C also means that while it in principle contains the most non-vanishing cuts, most can be related to each other by relabeling.

5.2 t-cuts

The t-cuts of the 6-point loop have a relatively simple structure; since a 6-point t-cut consists of a product of two 5-point trees, this always results in a product of an MHV tree amplitude and an $\overline{\text{MHV}}$ tree. As such we can solve the general n-point case of an MHV $\times$ $\overline{\text{MHV}}$ cut for arbitrary choice of the lone negative and positive leg, respectively, and apply this general result to solve all the 6-point t-cuts. We thus have the configuration

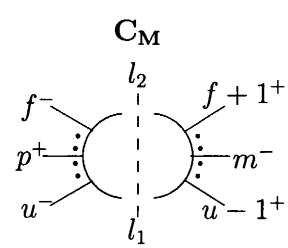


Figure 5.1: The general MHV $\times$ $\overline{\text{MHV}}$ t-cut

The l-dependent part of this cut coefficient is thus

$$C_M = \frac{\langle m\,l_2\rangle^2 \langle m\,l_1\rangle^2}{\langle f+1\,l_1\rangle \langle l_1\,l_2\rangle \langle l_2\,u-1\rangle \prod_{i=u+1}^{f-2} \langle i\,i+1\rangle} \times \frac{[p\,l_2]^2 [p\,l_1]^2}{[u\,l_2]\,[l_2\,l_1]\,[l_1\,f] \prod_{j=f}^{u-1} [j\,j+1]}. \tag{5.2.1}$$

This can be rewritten purely in terms of angle spinor products,

$$\begin{aligned} C_M =& \frac{[p|l_2|m\rangle^2 [p|l_1|m\rangle^2}{\langle f-1\,l_1\rangle \langle u+1\,l_2\rangle [u\,l_2]\,[f\,l_1]} \frac{1}{P^2 \prod_{i=u+1}^{f-2} \langle i\,i+1\rangle \prod_{j=f}^{u-1} [j\,j+1]}, \\ =& \frac{[p|l_1|m\rangle^2 [p|P|l_1\rangle^2}{P^2 \prod_{i=u+1}^{f-2} \langle i\,i+1\rangle \prod_{j=f}^{u-1} [j\,j+1] \langle f-1\,l_1\rangle [u|P|l_1\rangle} \frac{\langle l_2\,m\rangle^2}{\langle u+1\,l_2\rangle [f|P|l_2\rangle}. \end{aligned} \tag{5.2.2}$$

The problem of simplifying an H-function-like term with quadratic dependence on l and an arbitrary number of l_1 and l_2 dependent angle spinor product pairs is one which will be repeated frequently throughout the 6-point calculation; as such it is very helpful to consider how to reduce the general case of a quadratic H-function with an arbitrary number of l_2 spinor pairs to a function homogenized in l_1,

$$C_H = [A|l_1|B\rangle [C|l_1|D\rangle \prod_{i=1}^{m} \frac{\langle a_i\,l_1\rangle}{\langle b_i\,l_1\rangle} \prod_{j=1}^{n} \frac{\langle c_j\,l_2\rangle}{\langle d_j\,l_2\rangle}. \tag{5.2.3}$$

We split the l_2 dependent spinor product pairs using equation (3.2.9)

$$\begin{aligned} C_H =& [A|l_1|B\rangle [C|l_1|D\rangle \prod_{i=1}^{m} \frac{\langle a_i\,l_1\rangle}{\langle b_i\,l_1\rangle} \prod_{j=1}^{n} \left(\frac{\langle c_j\,l_1\rangle}{\langle d_j\,l_1\rangle} - \frac{\langle c_j\,d_j\rangle\,P^2}{\langle d_j\,l_1\rangle\,[l_1|P|d_j\rangle} \right), \\ =& [A|l_1|B\rangle [C|l_1|D\rangle \prod_{i=1}^{m} \frac{\langle a_i\,l_1\rangle}{\langle b_i\,l_1\rangle} \prod_{j=1}^{n} \frac{\langle c_j\,l_1\rangle}{\langle d_j\,l_1\rangle} \\ & - [A|l_1|B\rangle [C|l_1|D\rangle \prod_{i=1}^{m} \frac{\langle a_i\,l_1\rangle}{\langle b_i\,l_1\rangle} \sum_{k=1}^{n} \frac{\langle c_k\,d_k\rangle\,P^2}{\langle d_k\,l_1\rangle\,[l_1|P|d_k\rangle} \prod_{j=1,\neq k}^{n} \frac{\langle c_j\,l_1\rangle}{\langle d_j\,l_1\rangle} \\ & + [A|l_1|B\rangle [C|l_1|D\rangle \prod_{i=1}^{m} \frac{\langle a_i\,l_1\rangle}{\langle b_i\,l_1\rangle} \sum_{h=1}^{n} \sum_{k=1}^{h-1} \frac{P^2 \langle c_k\,d_k\rangle}{\langle d_k\,l_1\rangle\,[l_1|P|d_k\rangle} \frac{P^2 \langle c_h\,d_h\rangle}{\langle d_h\,l_1\rangle\,[l_1|P|d_h\rangle} \prod_{j=1,\neq k,h}^{n} \frac{\langle c_j\,l_1\rangle}{\langle d_j\,l_1\rangle}. \end{aligned} \tag{5.2.4}$$

We now convert the pairs of l_1 square products back into angle products,

$$
\begin{aligned}
=&[A|l_1|B\rangle[C|l_1|D\rangle \prod_{i=1}^{m} \frac{\langle a_i\, l_1\rangle}{\langle b_i\, l_1\rangle} \prod_{j=1}^{n} \frac{\langle c_j\, l_1\rangle}{\langle d_j\, l_1\rangle} \\
+&[A|l_1|B\rangle\, \langle l_1\, D\rangle \prod_{i=1}^{m} \frac{\langle a_i\, l_1\rangle}{\langle b_i\, l_1\rangle} \sum_{k=1}^{n} \frac{\langle c_k\, d_k\rangle\, [C|P|l_2\rangle}{\langle d_k\, l_1\rangle\, \langle d_k\, l_2\rangle} \prod_{j=1,\neq k}^{n} \frac{\langle c_j\, l_1\rangle}{\langle d_j\, l_1\rangle} \\
+&\langle l_1\, B\rangle\, \langle l_1\, D\rangle \prod_{i=1}^{m} \frac{\langle a_i\, l_1\rangle}{\langle b_i\, l_1\rangle} \sum_{h=1}^{n} \sum_{k=1}^{h-1} \frac{\langle c_k\, d_k\rangle\, \langle c_h\, d_h\rangle\, [A|P|l_2\rangle [C|P|l_2\rangle}{\langle d_k\, l_1\rangle\, \langle d_k\, l_2\rangle\, \langle d_h\, l_1\rangle\, \langle d_h\, l_2\rangle} \prod_{j=1,\neq k,h}^{n} \frac{\langle c_j\, l_1\rangle}{\langle d_j\, l_1\rangle}\,.
\end{aligned}
\tag{5.2.5}
$$

The first term is now dependent only upon l_1, and the last term is $O(l^0)$ and as such we can simply make the replacement $l_2 \to l_1$. The linear term however must be split again,

$$
\begin{aligned}
=&[A|l_1|B\rangle[C|l_1|D\rangle \prod_{i=1}^{m} \frac{\langle a_i\, l_1\rangle}{\langle b_i\, l_1\rangle} \prod_{j=1}^{n} \frac{\langle c_j\, l_1\rangle}{\langle d_j\, l_1\rangle} \\
+&[A|l_1|B\rangle\, \langle l_1\, D\rangle \prod_{i=1}^{m} \frac{\langle a_i\, l_1\rangle}{\langle b_i\, l_1\rangle} \sum_{k=1}^{n} \frac{\langle c_k\, d_k\rangle}{\langle d_k\, l_1\rangle} \left(\frac{[C|P|l_1\rangle}{\langle d_k\, l_1\rangle} - \frac{[C|P|d_k\rangle P^2}{\langle d_k\, l_1\rangle\, [l_1|P|d_k\rangle} \right) \prod_{j=1,\neq k}^{n} \frac{\langle c_j\, l_1\rangle}{\langle d_j\, l_1\rangle} \\
+&\langle l_1\, B\rangle\, \langle l_1\, D\rangle \prod_{i=1}^{m} \frac{\langle a_i\, l_1\rangle}{\langle b_i\, l_1\rangle} \sum_{h=1}^{n} \sum_{k=1}^{h-1} \frac{\langle c_k\, d_k\rangle\, \langle c_h\, d_h\rangle\, [A|P|l_1\rangle [C|P|l_1\rangle}{\langle d_k\, l_1\rangle^2\, \langle d_h\, l_1\rangle^2} \prod_{j=1,\neq k,h}^{n} \frac{\langle c_j\, l_1\rangle}{\langle d_j\, l_1\rangle}\,.
\end{aligned}
\tag{5.2.6}
$$

Tidying up these expressions we get

$$
\begin{aligned}
=&[A|l_1|B\rangle[C|l_1|D\rangle \prod_{i=1}^{m} \frac{\langle a_i\, l_1\rangle}{\langle b_i\, l_1\rangle} \prod_{j=1}^{n} \frac{\langle c_j\, l_1\rangle}{\langle d_j\, l_1\rangle} \\
&+ \sum_{k=1}^{n} \langle c_k\, d_k\rangle \, \frac{[A|l_1|B\rangle \, \langle D\, l_1\rangle \, [C|P|l_1\rangle}{\langle d_k\, l_1\rangle^2} \prod_{i=1}^{m} \frac{\langle a_i\, l_1\rangle}{\langle b_i\, l_1\rangle} \prod_{j=1, \neq k}^{n} \frac{\langle c_j\, l_1\rangle}{\langle d_j\, l_1\rangle} \\
&+ \sum_{k=1}^{n} \langle c_k\, d_k\rangle \, [C|P|d_k\rangle \frac{[A|P|l_1\rangle \, \langle B\, l_1\rangle \, [C|P|l_1\rangle \, \langle D\, l_1\rangle}{\langle d_k\, l_1\rangle^3} \prod_{i=1}^{m} \frac{\langle a_i\, l_1\rangle}{\langle b_i\, l_1\rangle} \prod_{j=1, \neq k}^{n} \frac{\langle c_j\, l_1\rangle}{\langle d_j\, l_1\rangle} \\
&+ \sum_{h=1}^{n} \sum_{k=1}^{h} \frac{\langle c_k\, d_k\rangle \, \langle c_h\, d_h\rangle \, [A|P|l_1\rangle [C|P|l_1\rangle \, \langle B\, l_1\rangle \, \langle D\, l_1\rangle}{\langle d_k\, l_1\rangle^2 \, \langle d_h\, l_1\rangle^2} \prod_{i=1}^{m} \frac{\langle a_i\, l_1\rangle}{\langle b_i\, l_1\rangle} \prod_{j=1, \neq k,h}^{n} \frac{\langle c_j\, l_1\rangle}{\langle d_j\, l_1\rangle}\, , \\
=&H^2_{\{i\}+\{j\}}(A, B, C, D; \{a_i\}, \{c_j\}; \{b_i\}, \{d_j\}; P) \\
&+ \sum_{k=1}^{n} \langle c_k\, d_k\rangle \, H^{1;x}_{1+\{i\}+\{j\setminus k\}}(A, B; [C|P, D, \{a_i\}, \{c_j \setminus c_k\}; \{b_i\}, \{d_j \setminus d_k\}; d_k; P) \\
&+ \sum_{k=1}^{n} \langle c_k\, d_k\rangle \, [C|P|d_k\rangle H^{0;xx}_{1+\{i\}+\{j\setminus k\}}([A|P, B, D, \{a_i\}, \{c_j \setminus c_k\}; \{b_i\}, \{d_j \setminus d_k\}; d_k; P) \\
&+ \sum_{h=1}^{n} \sum_{k=1}^{h-1} \langle c_k\, d_k\rangle \, \langle c_h\, d_h\rangle \\
&\times H^{0;xy}_{1+\{i\}+\{j\setminus k,h\}}([A|P, B, [C|P, D, \{a_i\}, \{c_j \setminus (c_k, c_h)\}; \{b_i\}, \{d_j \setminus (d_k, d_h)\}; d_k, d_h; P)\, .
\end{aligned}
\tag{5.2.7}
$$

This can be applied to the t-cut by making the identification,

$$
\begin{aligned}
&A = p,\ B = m,\ C = p,\ D = m,\ \{a_i\} = ([p|P, [p|P),\ \{b_i\} = (f-1, [u|P), \\
&\{c_j\} = (m, m),\ \{d_j\} = (u+1, [f|P)\, .
\end{aligned}
$$

With this prescription we obtain

$$
\begin{aligned}
&C_M(f, \cdots, u, u+1, \cdots, f-1; p, m, P) = \\
&\quad \frac{1}{P^2 \prod_{i=u+1}^{f-2} \langle i\, i+1\rangle \prod_{j=f}^{u-1} [j\, j+1]} \Big(H^2_4(p, m, p, m; [p|P, [p|P, m, m; f-1, [u|P, u+1, [f|P; P) \\
&\quad + \langle m\, u+1\rangle \, H^{1;x}_4(p, m; [p|P, m, [p|P, [p|P, m; f-1, [u|P, [f|P; u+1; P) \\
&\quad - [f|P|m\rangle H^{1;x}_4(p, m; [p|P, m, [p|P, [p|P, m; f-1, [u|P, u+1; [f|P; P) \\
&\quad + \langle m\, u+1\rangle \, [p|P|u+1\rangle H^{0;xx}_4([p|P, m, m, [p|P, [p|P, m; f-1, [u|P, [f|P; u+1; P) \\
&\quad + P^2 [f|P|m\rangle \, [p\, f] \, H^{0;xx}_4([p|P, m, m, [p|P, [p|P, m; f-1, [u|P, u+1; [f|P; P) \\
&\quad - \langle m\, u+1\rangle \, [f|P|m\rangle H^{0;xy}_3([p|P, m, [p|P, m, [p|P, [p|P; f-1, [u|P; u+1, [f|P; P)\Big)\, .
\end{aligned}
\tag{5.2.8}
$$

This general expression can thus be used to evaluate all t-cuts contributing to the 6-point amplitude,

$$c_2^A = C_M(2,3,4,5,6,1;4,1,P_{234}),\ c_3^A = C_M(6,1,2,3,4,5;6,3,P_{345}),$$

$$c_1^B = C_M(1,2,3,4,5,6;3,4,P_{123}),\ c_2^B = C_M(2,3,4,5,6,1;3,1,P_{234}),$$
$$c_3^B = C_M(6,1,2,3,4,5;6,4,P_{345}), \tag{5.2.9}$$

$$c_1^C = C_M(1,2,3,4,5,6;2,5,P_{123}),\ c_2^C = C_M(5,6,1,2,3,4;6,3,P_{234}),$$
$$c_3^C = C_M(3,4,5,6,1,2;4,1,P_{345}),$$

5.3 s-cuts

The four possible s-cut configurations as described in section 5.1 require the form of the 6-point NMHV tree, and cannot therefore be straightforwardly solved for the general case. The 6-point tree in the configuration required (four external gluons and two external scalars) was derived in 2006 by Bidder, Dunbar and Perkins [80], and when applied to the s-cuts typically results in a cut consisting of two quadratic H-function-like terms and one quadratic G-function term, with the exception of the split-helicity C_A cut which consists purely of H-functions. The approach followed here is to compute the H-function terms using the general formula (5.2.7), and to solve the three G-function terms individually.

5.3.1 D_A

The D_A function is the simplest of the configurations, due to depending upon the split-helicity NMHV 6-point tree,

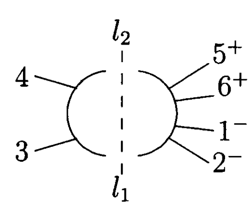


Figure 5.2: The D_A cut

This cut has the form

$$
\begin{aligned}
D_A(1,2,3,4,5,6;m) =& A_6^{tree}(l_1,l_2,5^+,6^+,1^-,2^-) \times A_4^{tree}(l_1,3,4,l_2)\,,\\
=& \Bigg(-\frac{[l_2|K_{2l_1l_2}|1\rangle[l_1|K_{2l_1l_2}|1\rangle^2}{t_{2l_1l_2}\,[2\,l_1]\,[l_1\,l_2]\,\langle 5\,6\rangle\,\langle 6\,1\rangle\,[2|K_{2l_1l_2}|5\rangle}\\
&+\frac{[6|K_{612}|l_1\rangle[6|K_{612}|l_2\rangle^2}{t_{612}\,[6\,1]\,[1\,2]\,\langle l_1\,l_2\rangle\,\langle l_2\,5\rangle\,[2|K_{612}|5\rangle}\Bigg) \times \frac{\langle m\,l_2\rangle^2\,\langle m\,l_1\rangle^2}{\langle 4\,l_2\rangle\,\langle l_2\,l_1\rangle\,\langle l_1\,3\rangle\,\langle 3\,4\rangle}\,.
\end{aligned}
\tag{5.3.1}
$$

The first term can be rewritten to depend only upon angle spinor product pairs,

$$
D_A^A = \frac{\langle m|l_1K_{56}|1\rangle^2\,\langle m\,l_2\rangle^2\,\langle l_1|P_{34}K_{56}|1\rangle}{t_{561}[2|P_{34}|l_2\rangle\,\langle 5\,6\rangle\,\langle 6\,1\rangle\,[2|K_{61}|5\rangle\,\langle 3\,4\rangle\,\langle 3\,l_1\rangle\,\langle 4\,l_2\rangle\,s_{34}}\,. \tag{5.3.2}
$$

This is in an appropriate form to apply equation (5.2.7), with the prescription

$$
A = \langle 1|K_{56}, B = m, C = \langle 1|K_{56}, D = m, a = \langle 1|K_{56}P_{34}, b = 3, \{c_j\} = (m,m), \{d_j\} = (4,[2|P_{34})\,,
$$

$$
\begin{aligned}
D_A^A =& \frac{1}{t_{561}s_{34}\,\langle 3\,4\rangle\,\langle 5\,6\rangle\,\langle 6\,1\rangle\,[2|K_{61}|5\rangle} \times \big(H_3^2(\langle 1|K_{56},m,\langle 1|K_{56},m;\langle 1|K_{56}P_{34},m,m;3,4,[2|P_{34};P_{34})\\
&+\langle m\,4\rangle\,H_3^{1;x}(\langle 1|K_{56},m;\langle 1|K_{56}P_{34},m,\langle 1|K_{56}P_{34},m;3,[2|P_{34};4;P_{34})\\
&-[2|P_{34}|m\rangle H_3^{1;x}(\langle 1|K_{56},m;\langle 1|K_{56}P_{34},m,\langle 1|K_{56}P_{34},m;3,4;[2|P_{34};P_{34})\\
&+\langle m\,4\rangle\,\langle 1|K_{56}P_{34}|4\rangle H_3^{0;xx}(\langle 1|K_{56}P_{34},m,m,\langle 1|K_{56}P_{34},m;3,[2|P_{34};4;P_{34})\\
&+[2|P_{34}|m\rangle[2|K_{56}|1\rangle s_{34}H_3^{0;xx}(\langle 1|K_{56}P_{34},m,m,\langle 1|K_{56}P_{34},m;3,4;[2|P_{34};P_{34})\\
&-\langle m\,4\rangle\,[2|P_{34}|m\rangle H_2^{0;xy}(\langle 1|K_{56}P_{34},m,\langle 1|K_{56}P_{34},m,\langle 1|K_{56}P_{34};3;4,[2|P_{34};P_{34})\big)\,.
\end{aligned}
\tag{5.3.3}
$$

The second term has the form

$$
D_A^B = \frac{[6|K_{12}P_{34}l_1|m\rangle^2[6|K_{12}|l_1\rangle\,\langle m\,l_2\rangle^2}{t_{612}s_{34}^2\,[6\,1]\,[1\,2]\,\langle 3\,4\rangle\,[2|K_{61}|5\rangle\,\langle 3\,l_1\rangle\,\langle 4\,l_2\rangle\,\langle 5\,l_2\rangle}\,, \tag{5.3.4}
$$

which with the prescription

$$
A = [6|K_{12}P_{34}, B = m, C = [6|K_{12}P_{34}, D = m, a = [6|K_{12}, b = 3, \{c_j\} = (m,m), \{d_j\} = (4,5),
$$

yields the expression

$$
\begin{aligned}
D_A^B = &- \frac{1}{t_{612} s_{34}^2 \, [6\,1]\, [1\,2]\, \langle 3\,4\rangle\, [2|K_{61}|5\rangle} \\
&\times \big(H_3^2([6|K_{12}P_{34}, m, [6|K_{12}P_{34}, m; [6|K_{12}, m, m; 3,4,5; P_{34}) \\
&+ s_{34} \langle m\,4\rangle\, H_3^{1;x}([6|K_{12}P_{34}, m; [6|K_{12}, m, [6|K_{12}, m; 3,5;4; P_{34}) \\
&+ s_{34} \langle m\,5\rangle\, H_3^{1;x}([6|K_{12}P_{34}, m; [6|K_{12}, m, [6|K_{12}, m; 3,4;5; P_{34}) \\
&+ s_{34}^2 \langle m\,4\rangle\, [6|K_{12}|4\rangle H_3^{0;xx}([6|K_{12}, m, m, [6|K_{12}, m; 3,5;4; P_{34}) \\
&+ s_{34}^2 \langle m\,5\rangle\, [6|K_{12}|5\rangle H_3^{0;xx}([6|K_{12}, m, m, [6|K_{12}, m; 3,4;5; P_{34}) \\
&+ s_{34}^2 \langle m\,4\rangle \langle m\,5\rangle\, H_2^{0;xy}([6|K_{12}, m, [6|K_{12}, m, [6|K_{12}; 3;4,5; P_{34}) \big) \,.
\end{aligned}
\tag{5.3.5}
$$

5.3.2 D_B

The D_B cut has the form

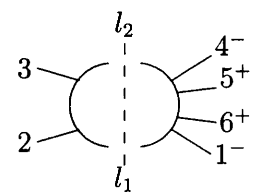


Figure 5.3: The D_B cut

This yields the cut

$$
\begin{aligned}
D_B(1,2,3,4,5,6;m) =& A_6^{tree}(l_1, l_2, 4^-, 5^+, 6^+, 1^-) \times A_4^{tree}(l_1, 2, 3, l_2)\,, \\
=& \Bigg(\frac{\langle 1\,l_1\rangle \langle 1\,l_2\rangle^2 [5\,6]^3}{t_{1l_1l_2} \langle l_1\,l_2\rangle [4\,5]\, [4|K_{1l_1l_2}|l_1\rangle [6|K_{1l_1l_2}|l_2\rangle} \\
&- \frac{[l_2|K_{l_1l_24}|1\rangle^2 [l_1|K_{l_1l_24}|1\rangle^2}{t_{l_1l_24} [l_1\,l_2]\, [l_2\,4] \langle 5\,6\rangle \langle 6\,1\rangle [l_1|K_{l_1l_24}|5\rangle [4|K_{l_1l_24}|1\rangle} \\
&+ \frac{[6|K_{61l_1}|4\rangle^2 [6\,l_1]^2 \langle l_2\,4\rangle^2}{t_{l_245} [6\,1]\, [1\,l_1] \langle l_2\,4\rangle \langle 4\,5\rangle [6|K_{61l_1}|l_2\rangle [l_1|K_{61l_1}|5\rangle} \Bigg) \times \frac{\langle m\,l_1\rangle^2 \langle m\,l_2\rangle^2}{\langle l_1\,2\rangle \langle 2\,3\rangle \langle 3\,l_2\rangle \langle l_2\,l_1\rangle}\,.
\end{aligned}
\tag{5.3.6}
$$

Unlike the D_A case the third term contains the factor t_{l_245} in the denominator, which will cause this term to give rise to G-function contributions. Tackling this term first, we get

$$
\begin{aligned}
D_B^C =& \frac{[6|P_{45l_2}|4\rangle^2\,[6\,l_1]^2\,\langle l_2\,4\rangle^2\,\langle m\,l_1\rangle^2\,\langle m\,l_2\rangle^2}{[6\,1]\,\langle 4\,5\rangle\,\langle 2\,3\rangle\,t_{l_2 45}\,\langle l_2\,4\rangle\,\langle 3\,l_2\rangle\,[1\,l_1]\,[6|K_{45}|l_2\rangle[l_1|K_{61}|5\rangle\,\langle l_1\,2\rangle\,\langle l_2\,l_1\rangle}\,,\\
=&\frac{1}{s_{23}\,\langle 2\,3\rangle\,\langle 4\,5\rangle\,[6\,1]}\\
&\times\frac{([6\,l_2]^2\langle l_2\,4\rangle^2+2\,[6\,l_2]\,\langle l_2\,4\rangle\,[6\,5]\,\langle 5\,4\rangle+[6\,5]^2\langle 5\,4\rangle^2)\langle m|P_{23}l_2|m\rangle[6|P_{23}|l_2\rangle^2\,\langle l_2\,4\rangle\langle m\,l_2\rangle\,\langle m\,l_1\rangle}{t_{l_2 45}[1|P_{23}|l_2\rangle[6|K_{45}|l_2\rangle\langle 5|K_{61}P_{23}|l_2\rangle\,\langle 3\,l_2\rangle\,\langle 2\,l_1\rangle}\,.
\end{aligned}
\tag{5.3.7}
$$

The bracket in the numerator will thus give rise to three separate terms, which can then be homogenized in l_2. The $\mathcal{O}(l^0)$ contribution is given by

$$
\begin{aligned}
D_B^{C1} =&\frac{[6\,5]^2\,\langle 5\,4\rangle^2\,\langle m|l_2P_{23}|m\rangle[6|P_{23}|l_2\rangle^2\,\langle l_2\,4\rangle\,\langle m\,l_2\rangle\,\langle m\,l_1\rangle}{\langle 2\,3\rangle\,\langle 4\,5\rangle\,[6\,1]\,s_{23}t_{l_2 45}[1|P_{23}|l_2\rangle[6|K_{45}|l_2\rangle\langle l_2|P_{23}K_{61}|5\rangle\,\langle 3\,l_2\rangle\,\langle 2\,l_1\rangle}\,,\\
=&-\frac{[6\,5]^2\,\langle 5\,4\rangle}{\langle 2\,3\rangle\,[6\,1]\,s_{23}}\\
&\times G_5^0(\langle m|P_{23},m;[6|P_{23},[6|P_{23},4,m,m;[1|P_{23},[6|K_{45},\langle 5|K_{61}P_{23},3,2;K_{45},P_{23},l_2)\,.
\end{aligned}
\tag{5.3.8}
$$

The linear term can be simply split using equation (3.2.9)

$$
\begin{aligned}
D_B^{C2} =&\frac{2\,[6\,5]\,\langle 5\,4\rangle\,[6\,l_2]\,\langle l_2\,4\rangle\,\langle m|l_2P_{23}|m\rangle[6|P_{23}|l_2\rangle^2\,\langle l_2\,4\rangle\,\langle m\,l_2\rangle\,\langle m\,l_1\rangle}{\langle 2\,3\rangle\,\langle 4\,5\rangle\,[6\,1]\,s_{23}t_{l_2 45}[1|P_{23}|l_2\rangle[6|K_{45}|l_2\rangle\langle l_2|P_{23}K_{61}|5\rangle\,\langle 3\,l_2\rangle\,\langle 2\,l_1\rangle}\,,\\
=&\frac{2\,[6\,5]\,\langle 5\,4\rangle\,[6\,l_2]\,\langle l_2\,4\rangle\,\langle m|l_2P_{23}|m\rangle[6|P_{23}|l_2\rangle^2\,\langle l_2\,4\rangle\,\langle m\,l_2\rangle\,\langle m\,l_2\rangle}{\langle 2\,3\rangle\,\langle 4\,5\rangle\,[6\,1]\,s_{23}t_{l_2 45}[1|P_{23}|l_2\rangle[6|K_{45}|l_2\rangle\langle l_2|P_{23}K_{61}|5\rangle\,\langle 3\,l_2\rangle\,\langle 2\,l_2\rangle}\\
&-\frac{2\,[6\,5]\,\langle 5\,4\rangle\,\langle m\,2\rangle\,\langle m|l_2P_{23}|m\rangle[6|P_{23}|l_2\rangle^3\,\langle l_2\,4\rangle^2\,\langle m\,l_2\rangle\,\langle m\,l_2\rangle}{\langle 2\,3\rangle\,\langle 4\,5\rangle\,[6\,1]\,s_{23}t_{l_2 45}[1|P_{23}|l_2\rangle[6|K_{45}|l_2\rangle\langle l_2|P_{23}K_{61}|5\rangle\,\langle 3\,l_2\rangle\,\langle 2\,l_2\rangle^2}\,,\\
=&-\frac{2\,[6\,5]}{s_{23}\,\langle 2\,3\rangle\,[6\,1]}\\
&\times G_5^1(\langle m|P_{23},m,6,4;[6|P_{23},[6|P_{23},4,m,m;[1|P_{23},[6|K_{45},\langle 5|K_{61}P_{23},3,2;K_{45},P_{23},l_2)\\
&-\frac{2\,[6\,5]\,\langle m\,2\rangle}{s_{23}\,\langle 2\,3\rangle\,[6\,1]}\\
&\times G_5^{0;x}(\langle m|P_{23},m;4,4,[6|P_{23},[6|P_{23},[6|P_{23},m;[1|P_{23},[6|K_{45},\langle 5|K_{61}P_{23},3;2;K_{45},P_{23},l_2)\,.
\end{aligned}
\tag{5.3.9}
$$

The hardest part is the quadratic, which requires two iterations of equation (3.2.9) to simplify,

$$
\begin{aligned}
D_B^{C3} =& \frac{[6\,l_2]^2 \langle l_2\,4\rangle^2 \langle m|P_{23}l_2|m\rangle [6|P_{23}|l_2\rangle^2 \langle l_2\,4\rangle \langle m\,l_2\rangle \langle m\,l_1\rangle}{s_{23}\langle 2\,3\rangle \langle 4\,5\rangle [6\,1]\, t_{l_2 45}[1|P_{23}|l_2\rangle [6|K_{45}|l_2\rangle \langle 5|K_{61}P_{23}|l_2\rangle \langle 3\,l_2\rangle \langle 2\,l_1\rangle}\,,\\
=& \frac{[6\,l_2]^2 \langle l_2\,4\rangle^2 \langle m|P_{23}l_2|m\rangle [6|P_{23}|l_2\rangle^2 \langle l_2\,4\rangle \langle m\,l_2\rangle^2}{s_{23}\langle 2\,3\rangle \langle 4\,5\rangle [6\,1]\, t_{l_2 45}[1|P_{23}|l_2\rangle [6|K_{45}|l_2\rangle \langle 5|K_{61}P_{23}|l_2\rangle \langle 3\,l_2\rangle \langle 2\,l_2\rangle}\\
&- \frac{[6\,l_2] \langle l_2\,4\rangle \langle m|P_{23}l_2|m\rangle [6|P_{23}|l_2\rangle^3 \langle l_2\,4\rangle^2 \langle m\,l_2\rangle \langle m\,2\rangle}{s_{23}\langle 2\,3\rangle \langle 4\,5\rangle [6\,1]\, t_{l_2 45}[1|P_{23}|l_2\rangle [6|K_{45}|l_2\rangle \langle 5|K_{61}P_{23}|l_2\rangle \langle 3\,l_2\rangle \langle 2\,l_2\rangle^2}\\
&+ \frac{\langle m|P_{23}l_2|m\rangle [6|P_{23}|l_2\rangle^3 \langle l_2\,4\rangle^3 \langle m\,l_2\rangle \langle m\,2\rangle [6|P_{23}|2\rangle}{s_{23}\langle 2\,3\rangle \langle 4\,5\rangle [6\,1]\, t_{l_2 45}[1|P_{23}|l_2\rangle [6|K_{45}|l_2\rangle \langle 5|K_{61}P_{23}|l_2\rangle \langle 3\,l_2\rangle \langle 2\,l_2\rangle^3}\,,\\
=& -\frac{1}{s_{23}\langle 2\,3\rangle \langle 4\,5\rangle [6\,1]}\\
&\times G_5^2(6,4,6,4,\langle m|P_{23},m;[6|P_{23},[6|P_{23},4,m,m;[1|P_{23},[6|K_{45},\langle 5|K_{61}P_{23},3,2;K_{45},P_{23},l_2)\\
&- \frac{\langle m\,2\rangle}{s_{23}\langle 2\,3\rangle \langle 4\,5\rangle [6\,1]}\\
&\times G_5^{1;x}(6,4,\langle m|P_{23},m;[6|P_{23},[6|P_{23},[6|P_{23},4,4,m;[1|P_{23},[6|K_{45},\langle 5|K_{61}P_{23},3;2;K_{45},P_{23},l_2)\\
&+ \frac{\langle m\,2\rangle [6|P_{23}|2\rangle}{s_{23}\langle 2\,3\rangle \langle 4\,5\rangle [6\,1]}\\
&\times G_5^{0;xx}(\langle m|P_{23},m;[6|P_{23},[6|P_{23},[6|P_{23},4,4,4,m;[1|P_{23},[6|K_{45},\langle 5|K_{61}P_{23},3;2;K_{45},P_{23},l_2)\,.
\end{aligned}
\tag{5.3.10}
$$

The D_B^A and D_B^B terms are now simple to evaluate using equation (5.2.7). The D_B^A term is given by

$$
\begin{aligned}
D_B^A =& \frac{\langle 1\,l_1\rangle^2 \langle 1\,l_2\rangle^2 [5\,6]^3 \langle m\,l_1\rangle^2 \langle m\,l_2\rangle^2}{t_{1l_1l_2}\langle l_1\,l_2\rangle [4\,5]\,[4|K_{1l_1l_2}|1\rangle [6|K_{1l_1l_2}|l_2\rangle \langle l_1\,2\rangle \langle 2\,3\rangle \langle 3\,l_2\rangle \langle l_2\,l_1\rangle}\,,\\
=& \frac{[5\,6]^3}{t_{456}s_{23}^2\langle 2\,3\rangle [4\,5]\,[4|K_{56}|1\rangle} \times \frac{\langle m|P_{23}l_1|m\rangle^2 \langle 1\,l_1\rangle \langle 1\,l_2\rangle^2}{\langle 2\,l_1\rangle \langle 3\,l_2\rangle [6|K_{45}|l_2\rangle}\,.
\end{aligned}
\tag{5.3.11}
$$

This can be homogenized in l_1 using equation (5.2.7), with the prescription

$$
A = \langle m|P_{23}, B = m, C = \langle m|P_{23}, D = m, a = 1, b = 2, \{c_j\} = (1,1), \{d_j\} = (3, [6|K_{45}),
$$

yielding

$$
\begin{aligned}
D_B^A =& \frac{[5\,6]^3}{t_{456}s_{23}^2\langle 2\,3\rangle [4\,5]\,[4|K_{56}|1\rangle} \times \big(H_3^2(\langle m|P_{23},m,\langle m|P_{23},m;1,1,1;2,3,[6|K_{45};P_{23})\\
&+ s_{23}\langle 1\,3\rangle\, H_3^{1;x}(\langle m|P_{23},m;m,m,1,1;2,[6|K_{45};3;P_{23})\\
&- s_{23}[6|K_{45}|1\rangle H_3^{1;x}(\langle m|P_{23},m;m,m,1,1;2,3;[6|K_{45};P_{23})\\
&+ s_{23}^2\langle 1\,3\rangle \langle m\,3\rangle\, H_3^{0;xx}(m,m,m,1,1;2,[6|K_{45};3;P_{23})\\
&+ s_{23}^2[6|K_{45}|1\rangle [6|K_{45}|m\rangle H_3^{0;xx}(m,m,m,1,1;2,3;[6|K_{45};P_{23})\\
&- s_{23}^2\langle 1\,3\rangle\, [6|K_{45}|1\rangle H_2^{0;xy}(m,m,m,m,1;2;3,[6|K_{45};P_{23})\big)\,.
\end{aligned}
\tag{5.3.12}
$$

The D_B^B is given by

$$
\begin{aligned}
D_B^B =& -\frac{[l_2|K_{l_1l_24}|1\rangle^2[l_1|K_{l_1l_24}|1\rangle^2}{t_{l_1l_24}\,[l_1\,l_2]\,[l_2\,4]\,\langle 5\,6\rangle\,\langle 6\,1\rangle\,[l_1|K_{l_1l_24}|5\rangle[4|K_{l_1l_24}|1\rangle} \times \frac{\langle m\,l_1\rangle^2\,\langle m\,l_2\rangle^2}{\langle l_1\,2\rangle\,\langle 2\,3\rangle\,\langle 3\,l_2\rangle\,\langle l_2\,l_1\rangle}\,, \\
=& \frac{1}{t_{561}s_{23}\,\langle 2\,3\rangle\,\langle 5\,6\rangle\,\langle 6\,1\rangle\,[4|K_{56}|1\rangle} \times \frac{\langle 1|K_{56}l_1|m\rangle^2\langle l_1|P_{23}K_{56}|1\rangle^2\,\langle m\,l_2\rangle^2}{\langle 2\,l_1\rangle\,\langle 3\,l_2\rangle\,[4|P_{23}|l_1\rangle\langle 5|K_{61}P_{23}|l_2\rangle}\,.
\end{aligned}
\tag{5.3.13}
$$

For this term the prescription is

$$
\begin{aligned}
&A = \langle 1|K_{56}, B = m, C = \langle 1|K_{56}, D = m, \{a_i\} = (\langle 1|K_{56}P_{23}, \langle 1|K_{56}P_{23}), \\
&\{b_i\} = (2, [4|P_{23}), \{c_j\} = (m, m), \{d_j\} = (3, \langle 5|K_{61}P_{23}),
\end{aligned}
$$

which gives

$$
\begin{aligned}
D_B^B =& -\frac{1}{t_{561}s_{23}\,\langle 2\,3\rangle\,\langle 5\,6\rangle\,\langle 6\,1\rangle\,[4|K_{56}|1\rangle} \\
&\times \Big(H_4^2(\langle 1|K_{56}, m, \langle 1|K_{56}, m; \langle 1|K_{56}P_{23}, \langle 1|K_{56}P_{23}, m, m; 2, [4|P_{23}, 3, \langle 5|K_{61}P_{23}; P_{23}) \\
&+ \langle m\,3\rangle\,H_4^{1;x}(\langle 1|K_{56}, m; \langle 1|K_{56}P_{23}, m, \langle 1|K_{56}P_{23}, \langle 1|K_{56}P_{23}, m; 2, [4|P_{23}, \langle 5|K_{61}P_{23}; 3; P_{23}) \\
&+\langle m|P_{23}K_{61}|5\rangle H_4^{1;x}(\langle 1|K_{56}, m; \langle 1|K_{56}P_{23}, m, \langle 1|K_{56}P_{23}, \langle 1|K_{56}P_{23}, m; 2, [4|P_{23}, 3; \langle 5|K_{61}P_{23}; P_{23}) \\
&+ \langle m\,3\rangle\,\langle 1|K_{56}P_{23}|3\rangle H_4^{0;xx}(\langle 1|K_{56}P_{23}, m, m, \langle 1|K_{56}P_{23}, \langle 1|K_{56}P_{23}, m; 2, [4|P_{23}, \langle 5|K_{61}P_{23}; 3; P_{23}) \\
&+\langle m|P_{23}K_{61}|5\rangle\,\langle 1\,5\rangle\,s_{23}t_{561} \\
&\times H_4^{0;xx}(\langle 1|K_{56}P_{23}, m, m, \langle 1|K_{56}P_{23}, \langle 1|K_{56}P_{23}, m; 2, [4|P_{23}, 3; \langle 5|K_{61}P_{23}; P_{23}) \\
&+ \langle m\,3\rangle\,\langle m|P_{23}K_{61}|5\rangle \\
&\times\, H_3^{0;xy}(\langle 1|K_{56}P_{23}, m, \langle 1|K_{56}P_{23}, m, \langle 1|K_{56}P_{23}, \langle 1|K_{56}P_{23}; 2, [4|P_{23}; 3, \langle 5|K_{61}P_{23}; P_{23})\Big)\,.
\end{aligned}
\tag{5.3.14}
$$

5.3.3 D_C

The D_C cut has the structure

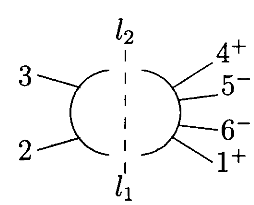


Figure 5.4: The D_C cut

$$D_C(1,2,3,4,5,6;m) = A_6(l_1,l_2,4^+,5^-,6^-,1^+) \times A_4(2,3,l_2,l_1)\,. \tag{5.3.15}$$

With the NMHV 6-point tree this gives the full expression

$$\begin{aligned} D_C = \Bigg(& \frac{[1\,l_1]\,[1\,l_2]^2\,\langle 5\,6\rangle^3}{t_{1l_1l_2}\,[l_1\,l_2]\,\langle 4\,5\rangle\,[1|K_{1l_1l_2}|4\rangle[l_2|K_{1l_1l_2}|6\rangle} \\ & - \frac{[1|K_{l_1l_24}|l_2\rangle^2[1|K_{l_1l_24}|l_1\rangle^2}{t_{l_1l_24}\,\langle l_1\,l_2\rangle\,\langle l_2\,4\rangle\,[5\,6]\,[6\,1]\,[5|K_{l_1l_24}|l_1\rangle[1|K_{l_1l_24}|4\rangle} \\ & + \frac{[4|K_{61l_1}|6\rangle^2\,\langle 6\,l_1\rangle^2\,[l_2\,4]^2}{t_{l_245}\,\langle 6\,1\rangle\,\langle 1\,l_1\rangle\,[l_2\,4]\,[4\,5]\,[l_2|K_{61l_1}|6\rangle[5|K_{61l_1}|l_1\rangle} \Bigg) \times \frac{\langle m\,l_1\rangle^2\,\langle m\,l_2\rangle^2}{\langle l_1\,2\rangle\,\langle 2\,3\rangle\,\langle 3\,l_2\rangle\,\langle l_2\,l_1\rangle}\,. \end{aligned} \tag{5.3.16}$$

Again, there is one G-function-containing term, which we will consider first,

$$\begin{aligned} D_C^G = & \frac{[4|K_{61l_1}|6\rangle^2\,\langle 6\,l_1\rangle^2\,[l_2\,4]^2}{t_{l_245}\,\langle 6\,1\rangle\,\langle 1\,l_1\rangle\,[l_2\,4]\,[4\,5]\,[l_2|K_{61l_1}|6\rangle[5|K_{61l_1}|l_1\rangle} \times \frac{\langle m\,l_1\rangle^2\,\langle m\,l_2\rangle^2}{\langle l_1\,2\rangle\,\langle 2\,3\rangle\,\langle 3\,l_2\rangle\,\langle l_2\,l_1\rangle}\,, \\ = & \frac{1}{s_{23}\,\langle 2\,3\rangle\,\langle 6\,1\rangle\,[4\,5]} \\ & \times \frac{([4\,1]^2\langle 1\,6\rangle^2 + 2\,[4\,1]\langle 1\,6\rangle[4\,l_1]\langle l_1\,6\rangle + [4\,l_1]^2\langle l_1\,6\rangle^2)\langle m|P_{23}l_1|m\rangle\langle m\,l_1\rangle[4|P_{23}|l_1\rangle\langle 6\,l_1\rangle^2\langle m\,l_2\rangle}{\langle 1\,l_1\rangle\,\langle 2\,l_1\rangle\,[5|K_{61}|l_1\rangle\langle 6|K_{45}P_{23}|l_1\rangle\,\langle 3\,l_2\rangle}\,. \end{aligned} \tag{5.3.17}$$

As with the D_B^G term, the numerator sum gives us a quadratic term, a linear term, and a term with overall power l^0. The latter term is trivial to homogenize in l_1 and is given by

$$\begin{aligned} D_C^{G1} = & \frac{[4\,1]^2\,\langle 1\,6\rangle^2}{s_{23}\,\langle 2\,3\rangle\,\langle 6\,1\rangle\,[4\,5]} \times \frac{\langle m|P_{23}l_1|m\rangle\,\langle m\,l_1\rangle\,[4|P_{23}|l_1\rangle\,\langle 6\,l_1\rangle^2\,\langle m\,l_1\rangle}{t_{l_161}\,\langle 1\,l_1\rangle\,[5|K_{61}|l_1\rangle\,\langle 2\,l_1\rangle\,\langle 6|K_{45}P_{23}|l_1\rangle\,\langle 3\,l_1\rangle}\,, \\ = & \frac{[4\,1]^2\,\langle 1\,6\rangle^2}{s_{23}\,\langle 2\,3\rangle\,\langle 6\,1\rangle\,[4\,5]} G_5^0(\langle m|P_{23},m;m,m,[4|P_{23},6,6;1,2,3,[5|K_{61},\langle 6|K_{45}P_{23};K_{61},P_{23},l_1)\,. \end{aligned} \tag{5.3.18}$$

The linear term can be homogenized with a single application of equation (3.2.9),

$$\begin{aligned} D_C^{G2} = & -\frac{2\,[4\,1]\,\langle 1\,6\rangle}{s_{23}\,\langle 6\,1\rangle\,\langle 2\,3\rangle\,[4\,5]} \times \frac{\langle m|P_{23}l_1|m\rangle\,[4\,l_1]\,\langle 6\,l_1\rangle\,\langle m\,l_1\rangle\,[4|P_{23}|l_1\rangle\,\langle 6\,l_1\rangle^2\,\langle m\,l_2\rangle}{t_{61l_1}\,\langle 1\,l_1\rangle\,\langle 2\,l_1\rangle\,[5|K_{61}|l_1\rangle\langle 6|K_{45}P_{23}|l_1\rangle\,\langle 3\,l_2\rangle}\,, \\ = & -\frac{2\,[4\,1]\,\langle 1\,6\rangle}{s_{23}\,\langle 6\,1\rangle\,\langle 2\,3\rangle\,[4\,5]} \\ & \times G_5^1(4,6,\langle m|P_{23},m;[4|P_{23},m,m,6,6;1,2,3,[5|K_{61},\langle 6|K_{45}P_{23};K_{61},P_{23},l_1) \\ & -\frac{2\,[4\,1]\,\langle 1\,6\rangle\,\langle m\,3\rangle}{s_{23}\,\langle 6\,1\rangle\,\langle 2\,3\rangle\,[4\,5]} \\ & \times G_5^{0;x}(\langle m|P_{23},m;m,[4|P_{23},[4|P_{23},6,6,6;1,2,[5|K_{61},\langle 6|K_{45}P_{23};3;K_{61},P_{23},l_1)\,. \end{aligned} \tag{5.3.19}$$

Again, the quadratic piece requires two iterations of equation (3.2.9)

$$
\begin{aligned}
D_C^{C3} =& \frac{[4|l_1|6\rangle^2 \langle m|P_{23}l_1|m\rangle \langle m\, l_1\rangle\, [4|P_{23}|l_1\rangle\, \langle 6\, l_1\rangle^2\, \langle m\, l_2\rangle}{s_{23}\,\langle 6\,1\rangle\,\langle 2\,3\rangle\,[4\,5]\,t_{61l_1}\,\langle 1\, l_1\rangle\,\langle 2\, l_1\rangle\,[5|K_{61}|l_1\rangle\langle 6|K_{45}P_{23}|l_1\rangle\,\langle 3\, l_2\rangle}\,, \\
=& -\frac{1}{s_{23}\,\langle 6\,1\rangle\,\langle 2\,3\rangle\,[4\,5]} \\
&\times G_5^2(4,6,4,6,\langle m|P_{23},m;m,m,6,6,[4|P_{23};1,2,3,[5|K_{61},\langle 6|K_{45}P_{23};K_{61},P_{23},l_1) \\
&+\frac{\langle m\,3\rangle}{s_{23}\,\langle 6\,1\rangle\,\langle 2\,3\rangle\,[4\,5]} \\
&\times G_5^{1;x}(4,6,\langle m|P_{23},m;m,6,6,6,[4|P_{23},[4|P_{23};1,2,[5|K_{61},\langle 6|K_{45}P_{23};3;K_{61},P_{23},l_1) \\
&-\frac{\langle m\,3\rangle\,[4|P_{23}|3\rangle}{s_{23}\,\langle 6\,1\rangle\,\langle 2\,3\rangle\,[4\,5]} \\
&\times G_5^{0;xx}((\langle m|P_{23},m;m,[4|P_{23},[4|P_{23},6,6,6,6;1,2,[5|K_{61},\langle 6|K_{45}P_{23};3;K_{61},P_{23},l_1)\,.
\end{aligned}
\tag{5.3.20}
$$

This leaves the simpler H-function-like D_C^A and D_C^B. Beginning with D_C^A,

$$
\begin{aligned}
D_C^A =& \frac{[1\, l_1]\,[1\, l_2]\,\langle 5\,6\rangle^3\,\langle m\, l_1\rangle^2\,\langle m\, l_2\rangle^2\,[1|P_{23}|l_1\rangle}{t_{456}s_{23}\,\langle 2\,3\rangle\,\langle 4\,5\rangle\,[1|K_{56}|4\rangle\,\langle l_1\,2\rangle\,\langle l_1|P_{23}K_{45}|6\rangle\,\langle 3\, l_2\rangle}\,, \\
=& \frac{\langle 5\,6\rangle^3}{t_{456}s_{23}\,\langle 2\,3\rangle\,\langle 4\,5\rangle\,[1|K_{56}|4\rangle}\,\frac{[1|l_1|m\rangle[1|l_2|m\rangle\,\langle m\, l_1\rangle\,[1|P_{23}|l_1\rangle\,\langle m\, l_2\rangle}{\langle 2\, l_1\rangle\,\langle 6|K_{45}P_{23}|l_1\rangle\,\langle 3\, l_2\rangle}\,, \\
=& \frac{\langle 5\,6\rangle^3}{t_{456}s_{23}\,\langle 2\,3\rangle\,\langle 4\,5\rangle\,[1|K_{56}|4\rangle}\,\frac{[1|l_1|m\rangle^2\,\langle m\, l_1\rangle\,[1|P_{23}|l_1\rangle\,\langle m\, l_2\rangle}{\langle 2\, l_1\rangle\,\langle 6|K_{45}P_{23}|l_1\rangle\,\langle 3\, l_2\rangle} \\
&+\frac{[1|P_{23}|m\rangle\,\langle 5\,6\rangle^3}{t_{456}s_{23}\,\langle 2\,3\rangle\,\langle 4\,5\rangle\,[1|K_{56}|4\rangle}\,\frac{[1|l_1|m\rangle\,\langle m\, l_1\rangle\,[1|P_{23}|l_1\rangle\,\langle m\, l_2\rangle}{\langle 2\, l_1\rangle\,\langle 6|K_{45}P_{23}|l_1\rangle\,\langle 3\, l_2\rangle}\,.
\end{aligned}
\tag{5.3.21}
$$

The quadratic term can be evaluated with the prescription for equation (5.2.7) of

$$
A=1, B=m, C=1, D=m, \{a_i\}=(m,[1|P_{23}), \{b_i\}=(2,\langle 6|K_{45}P_{23}), c=m, d=3,
$$

The full term is thus

$$
\begin{aligned}
D_C^A =& \frac{\langle 5\,6\rangle^3}{t_{456}s_{23}\,\langle 2\,3\rangle\,\langle 4\,5\rangle\,[1|K_{56}|4\rangle} \times \big(H_3^2(1,m,1,m;m,[1|P_{23},m;2,\langle 6|K_{45}P_{23},3;P_{23},l_1) \\
&+\langle m\,3\rangle\, H_3^{1;x}(1,m;[1|P_{23},m,m,[1|P_{23};2,\langle 6|K_{45}P_{23};3;P_{23},l_1) \\
&+\langle m\,3\rangle\,[1|P_{23}|3\rangle H_3^{0;xx}([1|P_{23},m,m,m,[1|P_{23};2,\langle 6|K_{45}P_{23};3;P_{23},l_1) \\
&+[1|P_{23}|m\rangle H_3^1(1,m;m,m,[1|P_{23};2,3,\langle 6|K_{45}P_{23};P_{23},l_1) \\
&-[1|P_{23}|m\rangle\,\langle m\,3\rangle\, H_3^{0;x}([1|P_{23},m,[1|P_{23},m;2,\langle 6|K_{45}P_{23};3;P_{23})\big)\,.
\end{aligned}
\tag{5.3.22}
$$

D_C^B is given by

$$
\begin{aligned}
D_C^B &= -\frac{[1|K_{56}|l_2\rangle^2[1|K_{56}|l_1\rangle^2\langle m\, l_1\rangle^2\langle m\, l_2\rangle^2}{t_{561}\langle l_1\, l_2\rangle^2[5\,6]\,[6\,1]\,\langle 2\,3\rangle\,[1|K_{56}|4\rangle\,\langle l_2\,4\rangle\,\langle l_1\,2\rangle\,\langle 3\, l_2\rangle\,[5|K_{61}|l_1\rangle}\,, \\
&= -\frac{1}{t_{561}s_{23}^2\langle 2\,3\rangle\,[5\,6]\,[6\,1]\,[1|K_{56}|4\rangle}\times\frac{[1|K_{56}P_{23}l_1|m\rangle^2[1|K_{56}|l_1\rangle^2\langle m\, l_2\rangle^2}{\langle 2\, l_1\rangle\,[5|K_{61}|l_1\rangle\,\langle 3\, l_2\rangle\,\langle 4\, l_2\rangle}\,.
\end{aligned}
\tag{5.3.23}
$$

This can be straightforwardly evaluated using equation (5.2.7) with

$$
\begin{aligned}
&A = [1|K_{56}P_{23}, B = m, C = [1|K_{56}P_{23}, D = m, \{a_i\} = ([1|K_{56}, [1|K_{56}),\\
&\{b_i\} = (2, [5|K_{61}), \{c_j\} = (m, m), \{d_j\} = (3, 4),
\end{aligned}
$$

which yields the result

$$
\begin{aligned}
D_C^B = &-\frac{1}{t_{561}s_{23}^2\langle 2\,3\rangle\,[5\,6]\,[6\,1]\,[1|K_{56}|4\rangle}\\
&\times\Big(H_4^2([1|K_{56}P_{23}, m, [1|K_{56}P_{23}, m; [1|K_{56}, [1|K_{56}, m, m; 2, [5|K_{61}; 3, 4; P_{23})\\
&+s_{23}\langle m\,3\rangle\, H_4^{1;x}([1|K_{56}P_{23}, m; [1|K_{56}, m, [1|K_{56}, [1|K_{56}, m; 2, [5|K_{61}, 4; 3; P_{23})\\
&+s_{23}\langle m\,4\rangle\, H_4^{1;x}([1|K_{56}P_{23}, m; [1|K_{56}, m, [1|K_{56}, [1|K_{56}, m; 2, [5|K_{61}, 3; 4; P_{23})\\
&+s_{23}^2\langle m\,3\rangle\,[1|K_{56}|3\rangle H_4^{0;xx}([1|K_{56}, m, m, [1|K_{56}, [1|K_{56}, m; 2, [5|K_{61}, 4; 3; P_{23})\\
&+s_{23}^2\langle m\,4\rangle\,[1|K_{56}|4\rangle H_4^{0;xx}([1|K_{56}, m, m, [1|K_{56}, [1|K_{56}, m; 2, [5|K_{61}, 3; 4; P_{23})\\
&+\, s_{23}^2\langle m\,3\rangle\,\langle m\,4\rangle\, H_3^{0;xy}([1|K_{56}, m, [1|K_{56}, m, [1|K_{56}, [1|K_{56}; 2, [5|K_{61}; 3, 4; P_{23})\Big)\,.
\end{aligned}
\tag{5.3.24}
$$

5.3.4 D_D

The full D_D cut is given by

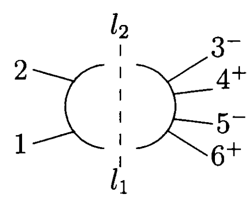


Figure 5.5: The D_D cut

$$
D_D(1,2,3,4,5,6;m) = A_6(l_1,l_2,3^-,4^+,5^-,6^+)\times A_4(1,2,l_2,l_1)\,. \tag{5.3.25}
$$

Inserting the expressions for the tree amplitudes, this gives us

$$\frac{\langle m\,l_1\rangle^2\,\langle m\,l_2\rangle^2}{\langle 1\,2\rangle\,\langle 2\,l_2\rangle\,\langle l_2\,l_1\rangle\,\langle l_1\,1\rangle}\times\left(-\frac{[l_2|K_{l_1l_23}|5\rangle^2[l_1|K_{l_1l_23}|5\rangle^2}{t_{123}\,[l_1\,l_2]\,[l_2\,3]\,\langle 4\,5\rangle\,\langle 5\,6\rangle\,[l_1|K_{l_1l_23}|4\rangle[3|K_{l_1l_23}|6\rangle}\right.$$
$$\left.-\frac{[6|K_{l_234}|3\rangle^2\,\langle l_2\,3\rangle\,[6\,l_1]^2}{t_{l_234}\,\langle l_2\,3\rangle\langle 3\,4\rangle[5\,6][6\,l_1][5|K_{l_234}|l_2\rangle[l_1|K_{l_234}|4\rangle}+\frac{[4|K_{345}|l_1\rangle^2[4|K_{345}|l_2\rangle^2}{t_{345}\langle 6\,l_1\rangle\langle l_1\,l_2\rangle[3\,4][4\,5][3|K_{345}|6\rangle[5|K_{345}|l_2\rangle}\right). \tag{5.3.26}$$

We thus have three terms to compute, two composed of H-functions and one (the second in the above expression) necessarily containing G-functions due to the presence of the factor t_{l_234}. We consider this term first,

$$D_D^B=\frac{[6|K_{l_234}|3\rangle^2\,\langle l_2\,3\rangle^2\,[6\,l_1]\,\langle l_2\,m\rangle^2\,\langle l_1\,m\rangle^2}{t_{l_234}\,\langle l_2\,3\rangle\,\langle 3\,4\rangle\,[5\,6]\,[5|K_{l_234}|l_2\rangle[l_1|K_{l_134}|4\rangle\,\langle 1\,2\rangle\,\langle 2\,l_2\rangle\,\langle l_2\,l_1\rangle\,\langle l_1\,1\rangle}. \tag{5.3.27}$$

Contracting the factor $\langle l_1\,l_2\rangle$ with the available square spinor product in the denominator, and expanding out the factor $[6|K_{l_234}|3\rangle$ in the numerator, we obtain

$$\begin{aligned}
D_D^B=&\frac{([6|l_2|3\rangle^2+2\,[6\,4]\,\langle 4\,3\rangle\,[6|l_2|3\rangle+[6\,4]^2\,\langle 4\,3\rangle^2)\,\langle l_2\,3\rangle^2\,[6\,l_1]\,\langle m\,l_2\rangle^2\,\langle m\,l_1\rangle^2}{\langle 3\,4\rangle\,[5\,6]\,\langle 1\,2\rangle\,t_{l_234}[5|K_{34}|l_2\rangle\langle 4|K_{56}P_{12}|l_2\rangle\,\langle 2\,l_2\rangle\,\langle l_1\,1\rangle\,\langle l_2\,3\rangle},\\
=&\frac{([6|l_2|3\rangle^2+2\,[6\,4]\,\langle 4\,3\rangle\,[6|l_2|3\rangle+[6\,4]^2\,\langle 4\,3\rangle^2)\,\langle l_2\,3\rangle^2\,\langle m\,l_2\rangle^2\,\langle m\,l_1\rangle\,[6|l_1|m\rangle}{\langle 3\,4\rangle\,[5\,6]\,\langle 1\,2\rangle\,t_{l_234}[5|K_{34}|l_2\rangle\langle 4|K_{56}P_{12}|l_2\rangle\,\langle 2\,l_2\rangle\,\langle l_1\,1\rangle\,\langle l_2\,3\rangle},\\
=&\frac{([6|l_2|3\rangle^2+2\,[6\,4]\,\langle 4\,3\rangle\,[6|l_2|3\rangle+[6\,4]^2\,\langle 4\,3\rangle^2)([6|l_2|m\rangle-[6|P|m\rangle)\,\langle l_2\,3\rangle^2\,\langle m\,l_2\rangle^2\,\langle m\,l_1\rangle}{\langle 3\,4\rangle\,[5\,6]\,\langle 1\,2\rangle\,t_{l_234}[5|K_{34}|l_2\rangle\langle 4|K_{56}P_{12}|l_2\rangle\,\langle 2\,l_2\rangle\,\langle 3\,l_2\rangle\,\langle l_1\,1\rangle}.
\end{aligned} \tag{5.3.28}$$

We will therefore have six terms arising from multiplying out the expressions in the numerator. Neglecting for now the constant prefactor $\frac{1}{\langle 3\,4\rangle[5\,6]\langle 1\,2\rangle}$ with the quadratic term, we have

$$D_D^{B_1}=\frac{[6|l_2|3\rangle^2\,\langle 3\,l_2\rangle^2\,\langle m\,l_2\rangle^2\,[6|l_2|m\rangle\,\langle m\,l_1\rangle}{t_{l_234}[5|K_{34}|l_2\rangle\langle 4|K_{56}P_{12}|l_2\rangle\,\langle 2\,l_2\rangle\,\langle 3\,l_2\rangle\,\langle 1\,l_1\rangle}. \tag{5.3.29}$$

We wish to write this entirely in terms of l_2, so we apply identity (3.2.9),

$$\begin{aligned}
D_D^{B_1}=&\frac{[6|l_2|3\rangle^2\,\langle 3\,l_2\rangle^2\,\langle m\,l_2\rangle^2\,[6|l_2|m\rangle}{t_{l_234}[5|K_{34}|l_2\rangle\langle 4|K_{56}P_{12}|l_2\rangle\,\langle 2\,l_2\rangle\,\langle 3\,l_2\rangle}\times\left(\frac{\langle m\,l_2\rangle}{\langle 1\,l_2\rangle}+\frac{P^2\,\langle m\,1\rangle}{\langle 1\,l_2\rangle\,[l_2|P_{12}|1\rangle}\right),\\
=&\frac{[6|l_2|3\rangle^2[6|l_2|m\rangle\,\langle 3\,l_2\rangle^2\,\langle m\,l_2\rangle^3}{t_{l_234}[5|K_{34}|l_2\rangle\langle 4|K_{56}P_{12}|l_2\rangle\,\langle 2\,l_2\rangle\,\langle 3\,l_2\rangle\,\langle 1\,l_2\rangle}\\
&+\frac{[6|l_2|3\rangle^2\,\langle 3\,l_2\rangle^2\,\langle m\,l_2\rangle^3\,\langle m\,1\rangle\,[6|P_{12}|l_1\rangle}{t_{l_234}[5|K_{34}|l_2\rangle\langle 4|K_{56}P_{12}|l_2\rangle\,\langle 1\,l_2\rangle\,\langle 2\,l_2\rangle\,\langle 3\,l_2\rangle\,\langle 1\,l_1\rangle}.
\end{aligned} \tag{5.3.30}$$

We thus have a quadratic term only dependent on l_2 and a linear term still containing l_1. Applying the identity again, we obtain

$$
\begin{aligned}
D_D^{B_1} =& \frac{[6|l_2|3\rangle^2[6|l_2|m\rangle\, \langle 3\, l_2\rangle^2\, \langle m\, l_2\rangle^3}{t_{l_2 34}[5|K_{34}|l_2\rangle\langle 4|K_{56}P_{12}|l_2\rangle\, \langle 2\, l_2\rangle\, \langle 3\, l_2\rangle\, \langle 1\, l_2\rangle} \\
&+ \frac{\langle m\, 1\rangle\, [6|l_2|3\rangle^2\, \langle 3\, l_2\rangle^2\, \langle m\, l_2\rangle^3\, [6|P_{12}|l_2\rangle}{t_{l_2 34}[5|K_{34}|l_2\rangle\langle 4|K_{56}P_{12}|l_2\rangle\, \langle 2\, l_2\rangle\, \langle 3\, l_2\rangle\, \langle 1\, l_2\rangle^2} \\
&+ \frac{\langle m\, 1\rangle\, [6|P_{12}|1\rangle[6|l_2|3\rangle\, \langle 3\, l_2\rangle^3\, \langle m\, l_2\rangle^3\, [6|P_{12}|l_2\rangle}{t_{l_2 34}[5|K_{34}|l_2\rangle\langle 4|K_{56}P_{12}|l_2\rangle\, \langle 2\, l_2\rangle\, \langle 3\, l_2\rangle\, \langle 1\, l_2\rangle^3},
\end{aligned}
\tag{5.3.31}
$$

where we have applied the equivalence $\frac{[6|P_{12}|l_1\rangle}{\langle 1\, l_1\rangle} \equiv \frac{[6|P_{12}|l_2\rangle}{\langle 1\, l_2\rangle}$ since it is of order l^0. The expression is now dependent only upon l_2 and we can insert our canonical forms,

$$
\begin{aligned}
D_D^{B_1} =& G_5^2(6,3,6,3,6,m;3,3,m,m,m;1,2,3,[5|K_{34},\langle 4|K_{56}P_{12},Q_{34},P_{12},l_2) \\
&+ \langle m\, 1\rangle\, G_5^{1;x}(6,3,6,3;3,3,m,m,m,[6|P_{12};2,3,[5|K_{34},\langle 4|K_{56}P_{12};1;Q_{34},P_{12},l_2) \\
&- \langle m\, 1\rangle\, [6|P_{12}|1\rangle G_5^{0;xx}(6,3;3,3,3,m,m,m,[6|P_{12};2,3,[5|K_{34},\langle 4|K_{56}P_{12};1;Q_{34},P_{12}) .
\end{aligned}
\tag{5.3.32}
$$

The first of the linear terms is given by

$$
D_D^{B_2} = \frac{[6|l_2|3\rangle^2\, \langle 3\, l_2\rangle^2\, \langle m\, l_2\rangle^2\, [6|P_{12}|m\rangle\, \langle m\, l_1\rangle}{t_{l_2 34}[5|K_{34}|l_2\rangle\langle 4|K_{56}P_{12}|l_2\rangle\, \langle 2\, l_2\rangle\, \langle 3\, l_2\rangle\, \langle 1\, l_1\rangle} . \tag{5.3.33}
$$

Applying the identity produces

$$
\begin{aligned}
D_D^{B_2} =& \frac{[6|l_2|3\rangle^2\, \langle 3\, l_2\rangle^2\, \langle m\, l_2\rangle^2\, [6|P_{12}|m\rangle}{t_{l_2 34}[5|K_{34}|l_2\rangle\langle 4|K_{56}P_{12}|l_2\rangle\, \langle 2\, l_2\rangle\, \langle 3\, l_2\rangle} \times \left(\frac{\langle m\, l_2\rangle}{\langle 1\, l_2\rangle} + \frac{P^2\, \langle m\, 1\rangle}{\langle 1\, l_2\rangle\, [l_2|P_{12}|1\rangle} \right) , \\
=& \frac{[6|P_{12}|m\rangle[6|l_2|3\rangle^2\, \langle 3\, l_2\rangle^2\, \langle m\, l_2\rangle^3}{t_{l_2 34}[5|K_{34}|l_2\rangle\langle 4|K_{56}P_{12}|l_2\rangle\, \langle 2\, l_2\rangle\, \langle 3\, l_2\rangle\, \langle 1\, l_2\rangle} \\
&+ \frac{[6|P_{12}|m\rangle\, \langle m\, 1\rangle\, [6|l_2|3\rangle\, \langle 3\, l_2\rangle^3\, \langle m\, l_2\rangle^2\, [6|P_{12}|l_2\rangle}{t_{l_2 34}[5|K_{34}|l_2\rangle\langle 4|K_{56}P_{12}|l_2\rangle\, \langle 2\, l_2\rangle\, \langle 3\, l_2\rangle\, \langle 1\, l_2\rangle\, \langle 1\, l_2\rangle} .
\end{aligned}
\tag{5.3.34}
$$

Again, we have been able to make the simple replacement $l_1 \longrightarrow l_2$ for the $O(l^0)$ term. Thus we have

$$
\begin{aligned}
D_D^{B_2} =& [6|P_{12}|m\rangle G_5^1(6,3,6,3;3,3,m,m,m;1,2,3,[5|K_{34},\langle 4|K_{56}P_{12};Q_{34},P_{12},l_2) \\
&+ [6|P_{12}|m\rangle\, \langle m\, 1\rangle\, G_5^{0;x}(6,3;3,3,3,m,m,[6|P_{12};2,3,[5|K_{34},\langle 4|K_{56}P_{12};1;Q_{34},P_{12}) .
\end{aligned}
\tag{5.3.35}
$$

The second linear term is given by

$$
D_D^{B_3} = \frac{2\,[6\, 4]\, \langle 4\, 3\rangle\, [6|l_2|3\rangle[6|l_2|m\rangle\, \langle 3\, l_2\rangle^2\, \langle m\, l_2\rangle^2\, \langle m\, l_1\rangle}{t_{l_2 34}[5|K_{34}|l_2\rangle\langle 4|K_{56}P_{12}|l_2\rangle\, \langle 2\, l_2\rangle\, \langle 3\, l_2\rangle\, \langle 1\, l_1\rangle} . \tag{5.3.36}
$$

This reduces to

$$\begin{aligned}D_D^{B_3} =& -2\,[6\,4]\,\langle 4\,3\rangle\, G_5^1(6,3,6,m;3,3,m,m,m;1,2,3,[5|K_{34},\langle 4|K_{56}P_{12};Q_{34},P_{12},l_2)\\ &+2\,[6\,4]\,\langle 4\,3\rangle\,\langle m\,1\rangle\, G_5^{0;x}(6,3;3,3,m,m,m,[6|P_{12};2,3,[5|K_{34},\langle 4|K_{56}P_{12};1;Q_{34},P_{12})\,.\end{aligned} \tag{5.3.37}$$

The two remaining terms are both $O(l^0)$, and thus trivially reduce to known canonical forms,

$$\begin{aligned}D_D^{B_4} =& \frac{2\,[6\,4]\,\langle 4\,3\rangle\,[6|P_{12}|m\rangle[6|l_2|3\rangle\,\langle l_2\,3\rangle^2\,\langle m\,l_2\rangle^2\,\langle m\,l_1\rangle}{t_{l_234}[5|K_{34}|l_2\rangle\langle 4|K_{56}P_{12}|l_2\rangle\,\langle 1\,l_2\rangle\,\langle 2\,l_2\rangle\,\langle 3\,l_2\rangle}\,,\\ =&2\,[6\,4]\,\langle 4\,3\rangle\,[6|P_{12}|m\rangle G_5^0(6,3;3,3,m,m,m;1,2,3,[5|K_{34},\langle 4|K_{56}P_{12};Q_{34},P_{12})\,,\end{aligned} \tag{5.3.38}$$

$$\begin{aligned}D_D^{B_5} =& \frac{[6\,4]^2\,\langle 4\,3\rangle^2\,[6|l_2|3\rangle\,\langle l_2\,3\rangle^2\,\langle m\,l_2\rangle^2\,\langle m\,l_1\rangle}{t_{l_234}[5|K_{34}|l_2\rangle\langle 4|K_{56}P_{12}|l_2\rangle\,\langle 1\,l_2\rangle\,\langle 2\,l_2\rangle\,\langle 3\,l_2\rangle}\,,\\ =& -[6\,4]^2\,\langle 4\,3\rangle^2\,G_5^0(6,m;3,3,m,m,m;1,2,3,[5|K_{34},\langle 4|K_{56}P_{12};Q_{34},P_{12})\,.\end{aligned} \tag{5.3.39}$$

In principle there is also term $D_D^{B_6}$, however it is of order l^{-1} and therefore does not contribute to the bubble coefficient.

The total D_D^B term of the D_D cut thus has the solution

$$\begin{aligned}D_D^B =& -\frac{1}{\langle 3\,4\rangle\,[5\,6]\,\langle 1\,2\rangle}\times\big(G_5^2(6,3,6,3,6,m;3,3,m,m,m;1,2,3,[5|K_{34},\langle 4|K_{56}P_{12},Q_{34},P_{12},l_2)\\ &+\langle m\,1\rangle\,G_5^{1;x}(6,3,6,3;3,3,m,m,m,[6|P_{12};2,3,[5|K_{34},\langle 4|K_{56}P_{12};1;Q_{34},P_{12},l_2)\\ &-\langle m\,1\rangle\,[6|P_{12}|1\rangle G_5^{0;xx}(6,3;3,3,3,m,m,m,[6|P_{12};2,3,[5|K_{34},\langle 4|K_{56}P_{12};1;Q_{34},P_{12})\\ &+[6|P_{12}|m\rangle G_5^1(6,3,6,3;3,3,m,m,m;1,2,3,[5|K_{34},\langle 4|K_{56}P_{12};Q_{34},P_{12},l_2)\\ &+[6|P_{12}|m\rangle\,\langle m\,1\rangle\,G_5^{0;x}(6,3;3,3,3,m,m,[6|P_{12};2,3,[5|K_{34},\langle 4|K_{56}P_{12};1;Q_{34},P_{12})\\ &-2\,[6\,4]\,\langle 4\,3\rangle\,G_5^1(6,3,6,m;3,3,m,m,m;1,2,3,[5|K_{34},\langle 4|K_{56}P_{12};Q_{34},P_{12},l_2)\\ &+2\,[6\,4]\,\langle 4\,3\rangle\,\langle m\,1\rangle\,G_5^{0;x}(6,3;3,3,m,m,m,[6|P_{12};2,3,[5|K_{34},\langle 4|K_{56}P_{12};1;Q_{34},P_{12})\\ &+2\,[6\,4]\,\langle 4\,3\rangle\,[6|P_{12}|m\rangle G_5^0(6,3;3,3,m,m,m;1,2,3,[5|K_{34},\langle 4|K_{56}P_{12};Q_{34},P_{12})\\ &-[6\,4]^2\,\langle 4\,3\rangle^2\,G_5^0(6,m;3,3,m,m,m;1,2,3,[5|K_{34},\langle 4|K_{56}P_{12};Q_{34},P_{12})\big)\,.\end{aligned} \tag{5.3.40}$$

The H-function terms can be evaluated using equation (5.2.7). Firstly the D_D^A term,

$$D_D^A = -\frac{\langle m\,l_1\rangle^2\,\langle m\,l_2\rangle^2}{\langle 1\,2\rangle\,\langle 2\,l_2\rangle\,\langle l_2\,l_1\rangle\,\langle l_1\,1\rangle}\times\frac{[l_2|K_{l_1l_23}|5\rangle^2[l_1|K_{l_1l_23}|5\rangle^2}{t_{123}\,[l_1\,l_2]\,[l_2\,3]\,\langle 4\,5\rangle\,\langle 5\,6\rangle\,[l_1|K_{l_1l_23}|4\rangle[3|K_{l_1l_23}|6\rangle}\,. \tag{5.3.41}$$

Tidying this up,

$$D_D^A = -\frac{1}{s_{12}t_{123}\,\langle 1\,2\rangle\,\langle 4\,5\rangle\,\langle 5\,6\rangle\,[3|K_{45}|6\rangle} \times \frac{[l_2|K_{46}|5\rangle^2[l_1|K_{46}|5\rangle^2\,\langle m\,l_1\rangle^2\,\langle m\,l_2\rangle^2}{\langle 2\,l_2\rangle\,\langle 1\,l_1\rangle\,[l_2\,3]\,[l_1|K_{56}|4\rangle}\,. \tag{5.3.42}$$

Multiplying top and bottom by $\langle l_1\,l_2\rangle$ to remove the square l dependent spinor products, we obtain

$$D_D^A = \frac{1}{s_{12}t_{123}\,\langle 1\,2\rangle\,\langle 4\,5\rangle\,\langle 5\,6\rangle\,[3|K_{45}|6\rangle} \times \frac{\langle 5|K_{46}P_{12}|l_1\rangle^2\langle m|l_1K_{46}|5\rangle^2\,\langle m\,l_2\rangle^2}{\langle 1\,l_1\rangle\,\langle 2\,l_2\rangle\,[3|P_{12}|l_1\rangle\langle 4|K_{56}P_{12}|l_2\rangle}\,. \tag{5.3.43}$$

This is now in a form where we can apply our general formula, with the parameters

$$m=2, n=2, A=\langle 5|K_{46}, B=m, C=\langle 5|K_{46}, D=m,$$
$$\{a_i\}=(\langle 5|K_{46}P_{12}, \langle 5|K_{46}P_{12}), \{b_i\}=(1,[3|P_{12}), \{c_j\}=(m,m), \{d_j\}=(2,\langle 4|K_{56}P_{12}),$$

This yields the expression

$$\begin{aligned}
D_D^A = &-\frac{1}{s_{12}t_{123}\,\langle 1\,2\rangle\,\langle 4\,5\rangle\,\langle 5\,6\rangle\,[3|K_{45}|6\rangle}\\
&\times \big(H_4^2(\langle 5|P_{46}, m, \langle 5|K_{46}, m; \langle 5|K_{46}P_{12}, \langle 5|K_{46}P_{12}, m, m; 1, [3|P_{12}, 2, \langle 4|K_{56}P_{12}; P_{12})\\
&+\langle m\,2\rangle\,H_4^{1;x}(\langle 5|K_{46}, m; \langle 5|K_{46}P_{12}, m, \langle 5|K_{46}P_{12}, \langle 5|K_{46}P_{12}, m; 1, [3|P_{12}, \langle 4|K_{56}P_{12}; 2; P_{12})\\
&+\langle m|P_{12}K_{56}|4\rangle H_4^{1;x}(\langle 5|K_{46}, m; \langle 5|K_{46}P_{12}, m, \langle 5|K_{46}P_{12}, \langle 5|K_{46}P_{12}, m; 1, [3|P_{12}, 2; \langle 4|K_{56}P_{12}; P_{12})\\
&+\langle m\,2\rangle\langle 5|K_{46}P_{12}|2\rangle H_4^{0;xx}(\langle 5|K_{46}P_{12}, m, m, \langle 5|K_{46}P_{12}, \langle 5|K_{46}P_{12}, m; 1, [3|P_{12}, \langle 4|K_{56}P_{12}; 2; P_{12})\\
&+\langle m|P_{12}K_{56}|4\rangle\,\langle 5\,4\rangle\,s_{12}t_{456}\\
&\times H_4^{0;xx}(\langle 5|K_{46}P_{12}, m, m, \langle 5|K_{46}P_{12}, \langle 5|K_{46}P_{12}, m; 1, [3|P_{12}, 2; \langle 4|K_{56}P_{12}; P_{12})\\
&+\langle m|P_{12}K_{56}|4\rangle\,\langle m\,2\rangle\\
&\times H_3^{0;xy}(\langle 5|K_{46}P_{12}, m, \langle 5|K_{46}P_{12}, m, \langle 5|K_{46}P_{12}, \langle 5|K_{46}P_{12}; 1, [3|P_{12}; 2, \langle 4|K_{56}P_{12}; P_{12})\big)\,.
\end{aligned} \tag{5.3.44}$$

The final part of the D_D cut is the D_D^C term,

$$D_D^C = \frac{\langle m\,l_1\rangle^2\,\langle m\,l_2\rangle^2}{\langle 1\,2\rangle\,\langle 2\,l_2\rangle\,\langle l_2\,l_1\rangle\,\langle l_1\,1\rangle} \times \frac{[4|K_{345}|l_1\rangle^2[4|K_{345}|l_2\rangle^2}{t_{345}\,\langle 6\,l_1\rangle\,\langle l_1\,l_2\rangle\,[3\,4]\,[4\,5]\,[3|K_{345}|6\rangle[5|K_{345}|l_2\rangle}\,. \tag{5.3.45}$$

This can be rewritten

$$
\begin{aligned}
D_D^C =& \frac{1}{t_{345}\langle 1\,2\rangle [3\,4]\,[4\,5]\,[3|K_{45}|6\rangle} \frac{[4|K_{35}|l_1\rangle^2[4|K_{35}|l_2\rangle^2 \langle m\,l_1\rangle^2 \langle m\,l_2\rangle^2}{\langle 1\,l_1\rangle \langle 6\,l_1\rangle \langle 2\,l_2\rangle [5|K_{34}|l_2\rangle \langle l_1\,l_2\rangle^2}, \\
=& \frac{1}{s_{12}^2 t_{345}\langle 1\,2\rangle [3\,4]\,[4\,5]\,[3|K_{45}|6\rangle} \frac{[4|K_{35}P_{12}l_1|m\rangle^2[4|K_{35}|l_1\rangle^2 \langle m\,l_2\rangle^2}{\langle 1\,l_1\rangle \langle 6\,l_1\rangle \langle 2\,l_2\rangle [5|K_{34}|l_2\rangle}.
\end{aligned}
\tag{5.3.46}
$$

Again we can apply the general formula, with the parameters

$$
\begin{aligned}
&n = 2, m = 2, A = [4|K_{35}P_{12}, B = m, C = [4|K_{35}P_{12}, D = m, \\
&\{a_i\} = ([4|K_{35}, [4|K_{35}), \{b_i\} = (1,6), \{c_j\} = (m,m), \{d_j\} = (2, [5|K_{34}),
\end{aligned}
$$

This gives a result for D_D^C of

$$
\begin{aligned}
D_D^C =& \frac{1}{s_{12}^2 t_{345}\langle 1\,2\rangle [3\,4]\,[4\,5]\,[3|K_{45}|6\rangle} \\
&\times \big(H_4^2([4|K_{35}P_{12}, m, [4|K_{35}P_{12}, m; [4|K_{35}, [4|K_{35}, m, m; 1, 6, 2, [5|K_{34}; P_{12}) \\
&+ s_{12}\langle m\,2\rangle H_4^{1;x}([4|K_{35}P_{12}, m; [4|K_{35}, m, [4|K_{35}, [4|K_{35}, m; 1, 6, [5|K_{34}; 2; P_{12}) \\
&- s_{12}[5|K_{34}|m\rangle H_4^{1;x}([4|K_{35}P_{12}, m; [4|K_{35}, m, [4|K_{35}, [4|K_{35}, m; 1, 6, 2, [5|K_{34}; P_{12}) \\
&+ s_{12}^2\langle m\,2\rangle [4|K_{35}|2\rangle H_4^{0;xx}([4|K_{35}, m, m, [4|K_{35}, [4|K_{35}, m; 1, 6, [5|K_{34}; 2; P_{12}) \\
&+ s_{12}^2 t_{345}\,[4\,5]\,[5|K_{34}|m\rangle H_4^{0;xx}([4|K_{35}, m, m, [4|K_{35}, [4|K_{35}, m; 1, 6, 2; [5|K_{34}; P_{12}) \\
&- s_{12}^2\langle m\,2\rangle [5|K_{34}|m\rangle H_3^{0;xy}([4|K_{35}, m, [4|K_{35}, m, [4|K_{35}, [4|K_{35}; 1, 6; 2, [5|K_{34}; P_{12})\big).
\end{aligned}
\tag{5.3.47}
$$

With the four D-cuts thus evaluated for the specific configurations given, one can apply them to solve all the s-cuts appearing in the 6-point scalar loop by relabeling,

$$
\begin{aligned}
&d_3^A = D_A(1,2,3,4,5,6;3),\ d_6^A = D_A(3,2,1,6,5,4;1), \\
\\
&d_2^B = D_B(1,2,3,4,5,6;2),\ d_3^B = D_A(3,4,5,6,1,2;4), \\
&d_4^B = D_C(3,4,5,6,1,2;4),\ d_6^B = D_D(6,1,2,3,4,5;1), \\
\\
&d_1^C = D_D(1,2,3,4,5,6;1),\ d_2^C = D_D(3,2,1,6,5,4;3), \\
&d_3^C = D_D(3,4,5,6,1,2;3),\ d_4^C = D_D(5,4,3,2,1,6;5), \\
&d_5^C = D_D(5,6,1,2,3,4;5),\ d_6^C = D_D(1,6,5,4,3,2;1).
\end{aligned}
\tag{5.3.48}
$$

With the full set of bubble coefficients computed, one can confirm the set of bubble coefficients is correct by confirming that they satisfy the necessary IR property that the coefficient of the $\frac{1}{\epsilon}$ part of the amplitude should sum to give the equivalent 6-point tree amplitude [84],

$$A_{IR}^{[0]} = \frac{c_\Gamma}{3\epsilon} A^{tree} . \tag{5.3.49}$$

This property is satisfied for the three partial amplitudes computed here. In addition, the three amplitudes can be evaluated at the kinematic point used in the semi-numerical calculation of the same amplitude [82],

$$\begin{aligned}
p_1 =& \frac{\mu}{2}(-1, +\sin\theta, +\cos\theta\sin\phi, +\cos\theta\cos\phi) ,\\
p_2 =& \frac{\mu}{2}(-1, -\sin\theta, -\cos\theta\sin\phi, -\cos\theta\cos\phi) ,\\
p_3 =& \frac{\mu}{3}(1, 1, 0, 0) ,\\
p_4 =& \frac{\mu}{7}(1, \cos\beta, \sin\beta, 0) ,\\
p_5 =& \frac{\mu}{6}(1, \cos\alpha\cos\beta, \cos\alpha\sin\beta, \sin\alpha) ,\\
p_6 =& -p_1 - p_2 - p_3 - p_4 - p_5 ,
\end{aligned} \tag{5.3.50}$$

where $\theta = \frac{\pi}{4}$, $\phi = \frac{\pi}{6}$, $\alpha = \frac{\pi}{3}$, $\cos\beta = -\frac{7}{19}$. In this case these results for the bubble coefficients agree numerically with the Ellis, Giele and Zanderighi estimate of the $\frac{1}{\epsilon}$ pole of the 6-point partial amplitude,

$---+++$	$0.3599 + i0.5782$
$--+-++$	$-0.7819 + i1.273$
$-+-+-+$	$-0.7475 - i0.7321$

5.4 Triangles and Boxes

As discussed in chapter 4, all three-mass triangle coefficients appearing at 7-point and below can be evaluated using a single, general triple cut with MHV corners, as shown in figure (4.3). The cut for the scalar case is

$$\begin{aligned}
&\frac{\langle m_1\,\ell_0\rangle^2\langle m_1\,\ell_1\rangle^2}{\langle \ell_0\,f_1\rangle\langle f_1\cdots u_1\rangle\langle u_1\,\ell_1\rangle\langle \ell_1\,\ell_0\rangle} \times \frac{\langle m_2\,\ell_1\rangle^2\langle m_2\,\ell_2\rangle^2}{\langle \ell_1\,f_2\rangle\langle f_2\cdots u_2\rangle\langle u_2\,\ell_2\rangle\langle \ell_2\,\ell_1\rangle} \times \frac{\langle m_3\,\ell_2\rangle^2\langle m_3\,\ell_0\rangle^2}{\langle \ell_2\,f_3\rangle\langle f_3\cdots u_3\rangle\langle u_3\,\ell_0\rangle\langle \ell_0\,\ell_2\rangle}\\
&= C_0 \times \frac{\langle m_1\,\ell_0\rangle^2\langle m_1\,\ell_1\rangle^2}{\langle \ell_0\,f_1\rangle\langle u_1\,\ell_1\rangle} \times \frac{\langle m_2\,\ell_1\rangle^2\langle m_2\,\ell_2\rangle^2}{\langle \ell_1\,f_2\rangle\langle f_2\,\ell_2\rangle} \times \frac{\langle m_3\,\ell_2\rangle^2\langle m_3\,\ell_0\rangle^2}{\langle \ell_2\,f_3\rangle\langle u_3\,\ell_0\rangle} \times \frac{[\ell_1\,\ell_2]^2}{\langle \ell_0|\ell_1\ell_2|\ell_0\rangle} ,\\
&= C_0 \times \frac{\langle m_1\,\ell_0\rangle^2\langle m_3\,\ell_0\rangle^2}{\langle \ell_0\,f_1\rangle\langle u_1\,\ell_0\rangle} \times \frac{\langle m_1\,\ell_1\rangle^2}{\langle \ell_1\,f_2\rangle\langle u_2\,\ell_1\rangle} \times \frac{\langle m_3\,\ell_2\rangle^2}{\langle f_3\,\ell_2\rangle\langle \ell_2\,f_3\rangle} \times \frac{\langle m_2\,\ell_2\rangle^2[\ell_1\,\ell_2]^2\langle m_2\,\ell_1\rangle^2}{\langle \ell_0|K_2K_3|\ell_0\rangle} ,\\
&= C_0 \times \frac{\prod_{y\in T_2}\langle \ell_0\,y\rangle}{\prod_{x\in S}\langle \ell_0\,x\rangle} \times \frac{\langle m_2\,\ell_2\rangle^2[\ell_1\,\ell_2]^2\langle m_2\,\ell_1\rangle^2}{\langle \ell_0|K_2K_3|\ell_0\rangle},
\end{aligned} \tag{5.4.1}$$

where

$$T_2 = \{|m_1\rangle, |m_1\rangle, |m_3\rangle, |m_3\rangle, K_1K_2|m_3\rangle, K_1K_2|m_3\rangle, K_3K_2|m_1\rangle, K_3K_2|m_1\rangle\}, \tag{5.4.2}$$

Splitting the pole and defining $T' = T_2 - \{m_1, m_1, m_3\}$, the cut integrand becomes,

$$
\begin{aligned}
&C_0 \frac{\prod_{y\in T_2} \langle \ell_0\, y\rangle}{\prod_{x\in S} \langle \ell_0\, x\rangle} \times \frac{\langle m_2|\ell_1\ell_2|m_2\rangle^2}{\langle \ell_0|K_2K_3|\ell_0\rangle} \\
=&C_0 \sum_{|x\rangle\in S} C_x \frac{\langle \ell_0\, m_1\rangle\,\langle \ell_0\, m_1\rangle\,\langle \ell_0\, m_3\rangle}{\langle \ell_0\, x\rangle} \times \frac{(\langle m_2|K_1K_3|m_2\rangle + \langle m_2|\ell_0 K_2|m_2\rangle)^2}{\langle \ell_0|K_2K_3|\ell_0\rangle} \\
\rightarrow& b^{3m}_{3,s}(K_1,K_2,K_3,m_1,m_2,m_3) = C_0 \sum_{|x\rangle\in S} C_x\Big(\langle m_2|K_1K_3|m_2\rangle^2 J^0_1(x;m_1,m_1,m_3) \\
&\qquad + 2\langle m_2|K_1K_3|m_2\rangle J^1_1(x;m_2,m_1,m_1,m_3;K_2|m_2\rangle) \\
&\qquad + J^2_1(x;m_2,m_2,m_1,m_1,m_3;K_2|m_2\rangle,K_2|m_2\rangle)\Big),
\end{aligned}
\tag{5.4.3}
$$

where,

$$
C_x = \frac{\prod_{y\in T'} \langle y\, x\rangle}{\prod_{z\in S-\{x\}} \langle z\, x\rangle}, \tag{5.4.4}
$$

With this general expression for the scalar triple cut, we can obtain the triangle coefficients appearing in the 6-point amplitude,

$$
\begin{aligned}
b^B_1 = b^{3m}_{3,s}(K_{23},K_{45},K_{61},2,4,1),\ b^C_1 = b^{3m}_{3,s}(K_{12},K_{34},K_{56},1,3,5),& \\
b^C_2 = b^{3m}_{3,s}(K_{23},K_{45},K_{61},3,5,1)&
\end{aligned}
\tag{5.4.5}
$$

The boxes for the six gluon amplitudes were previously derived by Bidder, Bjerrum-Bohr, Dunbar and Perkins [81]. Since they can be trivially obtained by inserting the box solutions in appendix (C) the calculation will not be repeated here.

5.5 The 7-Gluon Scalar Loop

One of the key advantages of the canonical basis approach is the fact that one can easily apply canonical forms derived to solve cuts in one amplitude to quickly solve cuts of the same form appearing in a different amplitude. To illustrate this let us consider the case of the 7-gluon NMHV 1-loop amplitude with a complex scalar circulating in the loop. As with the 7-gluon $\mathcal{N}=1$ calculation, there are 4 possible configurations,

$$
\begin{aligned}
A &:(1^-,2^-,3^-,4^+,5^+,6^+,7^+), \\
B &:(1^-,2^-,3^+,4^-,5^+,6^+,7^+), \\
C &:(1^-,2^-,3^+,4^+,5^-,6^+,7^+), \\
D &:(1^-,2^+,3^-,4^+,5^-,6^+,7^+).
\end{aligned}
\tag{5.5.1}
$$

To see how the canonical forms from the 6-gluon calculation can be quickly reapplied to the 7-point we will consider the bubble coefficients of the simplest case, the split-helicity A configuration, which is solved by the n-point expression derived in [22]. This has only six non-zero bubble coefficients,

$$
\begin{aligned}
A_7^{1-loop}(1^-,2^-,3^-,4^+,5^+,6^+,7^+) =& a_2^A I_{2\{345\}6\{71\}}^{2me} + a_5^A I_{5\{671\}2\{34\}}^{2me} \\
&+c_2^A I_{234}^2 + c_3^A I_{345}^2 + c_6^A I_{671}^2 + c_7^A I_{712}^2 \\
&+d_3^A I_{34}^2 + d_7^A I_{71}^2 + \mathcal{R}\,.
\end{aligned}
\tag{5.5.2}
$$

5.5.1 7-Point C-Functions

Obtaining the 7-point C-functions is particularly straightforward now the 6-point D-functions have been derived. This is because the 7-point C-functions differ from the 6-point D-functions only by the insertion of an additional leg on the MHV tree side, and a relabeling. Since the 6-point cuts have been solved for arbitrary helicity configuration on the MHV side, the extra leg has no effect on the loop momentum dependence - it simply enters as an additional constant prefactor.

Two different t-cuts contribute to the 7-point split helicity configuration: The scalar analogues of the C_M and C_A functions from Chapter 4.2. The first of these is simply a special case of the general MHV $\times$ $\overline{\text{MHV}}$ cut derived in section 5.2. The C_A function meanwhile can be obtained from the 6-point scalar D_A cut given in section 5.2 by inserting an additional positive leg into the MHV side,

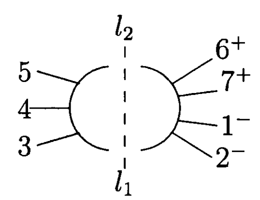


Figure 5.6: The 7-point C_A cut

$$
\begin{aligned}
C_A^A(1,2,3,4,5,6,7,m) = &\frac{1}{t_{671}t_{345}\langle 3\,4\rangle\langle 4\,5\rangle\langle 6\,7\rangle\langle 7\,1\rangle[2|K_{71}|6\rangle} \\
&\times\Big(H_3^2(\langle 1|K_{67},m,\langle 1|K_{67},m;\langle 1|K_{67}P_{345},m,m;3,5,[2|P_{345};P_{345}) \\
&+\langle m\,5\rangle\, H_3^{1;x}(\langle 1|K_{67},m;\langle 1|K_{67}P_{345},m,\langle 1|K_{67}P_{345},m;3,[2|P_{345};5;P_{345}) \\
&-[2|P_{345}|m\rangle H_3^{1;x}(\langle 1|K_{67},m;\langle 1|K_{67}P_{345},m,\langle 1|K_{67}P_{345},m;3,5;[2|P_{345};P_{345}) \\
&+\langle m\,5\rangle\,\langle 1|K_{67}P_{345}|5\rangle H_3^{0;xx}(\langle 1|K_{67}P_{345},m,m,\langle 1|K_{67}P_{345},m;3,[2|P_{345};5;P_{345}) \\
&+[2|P_{345}|m\rangle[2|K_{67}|1\rangle t_{345}H_3^{0;xx}(\langle 1|K_{67}P_{345},m,m,\langle 1|K_{67}P_{345},m;3,5;[2|P_{345};P_{345}) \\
&-\langle m\,5\rangle\,[2|P_{345}|m\rangle H_2^{0;xy}(\langle 1|K_{67}P_{345},m,\langle 1|K_{67}P_{345},m,\langle 1|K_{67}P_{345};3;5,[2|P_{345};P_{345})\Big)\,,
\end{aligned} \tag{5.5.3}
$$

$$
\begin{aligned}
C_A^B(1,2,3,4,5,6,7,m,P) = -&\frac{1}{t_{712}t_{345}^2\,[7\,1]\,[1\,2]\,\langle 3\,4\rangle\langle 4\,5\rangle[2|K_{71}|6\rangle} \\
&\times\Big(H_3^2([7|K_{12}P_{345},m,[7|K_{12}P_{345},m;[7|K_{12},m,m;3,5,6;P_{345}) \\
&+t_{345}\langle m\,5\rangle\, H_3^{1;x}([7|K_{12}P_{345},m;[7|K_{12},m,[7|K_{12},m;3,6;5;P_{345}) \\
&+t_{345}\langle m\,6\rangle\, H_3^{1;x}([7|K_{12}P_{345},m;[7|K_{12},m,[7|K_{12},m;3,5;6;P_{345}) \\
&+t_{345}^2\langle m\,5\rangle\,[7|K_{12}|5\rangle H_3^{0;xx}([7|K_{12},m,m,[7|K_{12},m;3,6;5;P_{345}) \\
&+t_{345}^2\langle m\,6\rangle\,[7|K_{12}|6\rangle H_3^{0;xx}([7|K_{12},m,m,[7|K_{12},m;3,5;6;P_{345}) \\
&+t_{345}^2\langle m\,5\rangle\,\langle m\,6\rangle\, H_2^{0;xy}([7|K_{12},m,[7|K_{12},m,[7|K_{12};3;5,6;P_{345})\Big)\,.
\end{aligned} \tag{5.5.4}
$$

The t-cuts of the A amplitude are thus given by

$$
\begin{aligned}
c_2^A &= C_M(2,3,4,5,6,7,1;4,1,P_{234}),\; c_3^A = C_A(1,2,3,4,5,6,7;3,P_{234}), \\
c_6^A &= -C_A(3,2,1,7,6,5,4;1,P_{671}),\; c_7^A = C_M(7,1,2,3,4,5,6;7,3,P_{712}).
\end{aligned} \tag{5.5.5}
$$

5.5.2 7-Point D_A-Function

The scalar analogue of the D_A derived in Chapter 4 has the form

$$
D_A(1,2,3^-,4^-,5^+,6^+,7^+) = A_4^{tree}(-l_1,1,2,l_2)\times A_7^{tree}(-l_2,3^-,4^-,5^+,6^+,7^+,l_1)\,. \tag{5.5.6}
$$

There are four terms in the cut, corresponding to the four parts of the 7-point tree. The first is given by

$$
\begin{aligned}
D_A^{1a} =& \frac{\langle m\,l_1\rangle^2\langle m\,l_2\rangle^2}{\langle 1\,2\rangle\langle 2\,l_2\rangle\langle l_2\,l_1\rangle\langle l_1\,1\rangle}\frac{[5|K_{34}|l_2\rangle[5|K_{34}|l_1\rangle^2}{\langle 6\,7\rangle\langle 7\,l_1\rangle\langle l_1\,l_2\rangle[3\,4]\,[4\,5]\,[3|K_{45}|6\rangle t_{345}}\,, \\
=& \frac{1}{\langle 1\,2\rangle\langle 6\,7\rangle[3\,4]\,[4\,5]\,[3|K_{45}|6\rangle t_{345}s_{12}^2}\frac{\langle m|P_{12}l_1|m\rangle^2[5|K_{34}|l_1\rangle^2[5|K_{34}|l_2\rangle}{\langle 1\,l_1\rangle\langle 7\,l_1\rangle\langle 2\,l_2\rangle}\,.
\end{aligned} \tag{5.5.7}
$$

We apply our general formula for homogenizing in l_1, with the choices

$$A = \langle m|P_{12}, B = m, C = \langle m|P_{12}, D = m, a_i = ([5|K_{34}, [5|K_{34}), b_i = (1,7), c_j = [5|K_{34}, d_j = 2,$$

which yields

$$\begin{aligned}
D_A^{1a} =& \frac{1}{\langle 1\,2\rangle \langle 6\,7\rangle [3\,4]\,[4\,5]\,[3|K_{45}|6\rangle t_{345} s_{12}^2} \\
&\times (H_3^2(\langle m|P_{12}, m, \langle m|P_{12}, m; [5|K_{34}, [5|K_{34}, [5|K_{34}; 1, 7, 2; P_{12}) \\
&+ s_{12}[5|K_{34}|2\rangle H_3^{1;x}(\langle m|P_{12}, m; m, m, [5|K_{34}, [5|K_{34}; 1, 7; 2; P_{12}) \\
&+ s_{12}^2[5|K_{34}|2\rangle \langle m\,2\rangle H_3^{0;xx}(m, m, m, [5|K_{34}, [5|K_{34}; 1, 7; 2; P_{12}))\,.
\end{aligned} \tag{5.5.8}$$

D_A^{1b} is the most difficult term and will be considered last. D_A^2 is straightforward and is given by

$$\begin{aligned}
D_A^2 =& -\frac{\langle m\,l_1\rangle^2 \langle m\,l_2\rangle^2}{\langle 1\,2\rangle \langle 2\,l_2\rangle \langle l_2\,l_1\rangle \langle l_1\,1\rangle} \frac{[l_2\,7]^2\,[l_1\,7]\,\langle 3\,4\rangle^4}{[l_1\,l_2]\,\langle 3\,4\rangle \langle 4\,5\rangle \langle 5\,6\rangle\,[l_2|K_{7l_1l_2}|6\rangle [7|K_{7l_1l_2}|3\rangle t_{7l_1l_2}}\,, \\
=& \frac{\langle 3\,4\rangle^4}{s_{12}\,\langle 3\,4\rangle \langle 4\,5\rangle \langle 5\,6\rangle\,[7|K_{456}|3\rangle t_{712}} \frac{[7|l_1|m\rangle [7|l_2|m\rangle [7|P_{12}|l_1\rangle \langle m\,l_1\rangle \langle m\,l_2\rangle}{\langle 2\,l_2\rangle \langle 1\,l_1\rangle \langle 6|K_{345}P_{12}|l_1\rangle}\,, \\
=& \frac{\langle 3\,4\rangle^4}{s_{12}\,\langle 1\,2\rangle \langle 3\,4\rangle \langle 4\,5\rangle \langle 5\,6\rangle\,[7|K_{456}|3\rangle t_{712}} \Big(\frac{[7|l_1|m\rangle^2 [7|P_{12}|l_1\rangle \langle m\,l_1\rangle \langle m\,l_2\rangle}{\langle 2\,l_2\rangle \langle 1\,l_1\rangle \langle 6|K_{345}P_{12}|l_1\rangle} \\
&+ \frac{[7|P_{12}|m\rangle [7|l_1|m\rangle [7|P_{12}|l_1\rangle \langle m\,l_1\rangle \langle m\,l_2\rangle}{\langle 2\,l_2\rangle \langle 1\,l_1\rangle \langle 6|K_{345}P_{12}|l_1\rangle} \Big)\,.
\end{aligned} \tag{5.5.9}$$

We apply the formula to the first part with the parametrization

$$A = 7, B = m, C = 7, D = m, a_i = ([7|P_{12}, m), b_i = (1, \langle 6|K_{345}P_{12}), c_j = m, d_j = 2,$$

while the second part is linear and fairly straightforward, yielding a total term of

$$\begin{aligned}
D_A^2 =& \frac{\langle 3\,4\rangle^4}{s_{12} t_{712}\,\langle 1\,2\rangle \langle 3\,4\rangle \langle 4\,5\rangle \langle 5\,6\rangle\,[7|K_{456}|3\rangle} \\
&\times (H_3^2(7, m, 7, m; [7|P_{12}, m, m; 1, \langle 6|K_{345}P_{12}, 2; P_{12}) \\
&+ \langle m\,2\rangle H_3^{1;x}(7, m; [7|P_{12}, m, [7|P_{12}, m; 1, \langle 6|K_{345}P_{12}; 2; P_{12}) \\
&+ \langle m\,2\rangle\,[7|P_{12}|m\rangle H_3^{0;xx}([7|P_{12}, m, m, [7|P_{12}, m; 1, \langle 6|K_{345}P_{12}; 2; P_{12})) \\
&+ \frac{\langle 3\,4\rangle^4\,[7|P_{12}|m\rangle}{s_{12} t_{712}\,\langle 1\,2\rangle \langle 3\,4\rangle \langle 4\,5\rangle \langle 5\,6\rangle\,[7|K_{456}|3\rangle} \\
&\times (H_3^1(7, m; [7|P_{12}, m, m; 1, \langle 6|K_{345}P_{12}, 2; P_{12}) \\
&- \langle m\,2\rangle H_3^{0;x}([7|P_{12}, m, [7|P_{12}, m; 1, \langle 6|K_{345}P_{12}; 2; P_{12}))\,.
\end{aligned} \tag{5.5.10}$$

D_A^3 is given by

$$D_A^3 = \frac{\langle m\, l_1\rangle^2 \langle m\, l_2\rangle^2}{\langle 1\,2\rangle\langle 2\, l_2\rangle\langle l_2\, l_1\rangle\langle l_1\, 1\rangle} \frac{\langle l_1\, 3\rangle^2 \langle l_2\, 3\rangle^2 [7|K_{l_1 l_2 3}|4\rangle^4}{\langle l_1\, l_2\rangle\langle l_2\, 3\rangle\langle 4\,5\rangle\langle 5\,6\rangle[7|K_{l_1 l_2 3}|3\rangle[7|K_{l_1 l_2 3}|4\rangle\langle l_1|K_{l_2 3}K_{45}|6\rangle t_{l_1 l_2 3} t_{456}},$$
$$= \frac{[7|K_{56}|4\rangle^4}{s_{12}^2 t_{123} t_{456}\,\langle 1\,2\rangle\,\langle 4\,5\rangle\,\langle 5\,6\rangle\,[7|K_{456}|3\rangle[7|K_{56}|4\rangle} \frac{\langle m|P_{12} l_1|m\rangle^2\,\langle l_1\, 3\rangle^2\,\langle l_2\, 3\rangle^2}{\langle 2\, l_2\rangle\,\langle 1\, l_1\rangle\,\langle 3\, l_2\rangle\,\langle 6|K_{45}K_{123}|l_1\rangle}. \tag{5.5.11}$$

We can solve this using the formula; our prescription is

$$A = \langle m|P_{12}, B = m, C = \langle m|P_{12}, D = m, a_i = (3,3), b_i = (1, \langle 6|K_{45}K_{123}), c_j = 3, d_j = 2,$$

which gives us

$$\begin{aligned} D_A^3 =& \frac{[7|K_{56}|4\rangle^4}{s_{12}^2 t_{123} t_{456}\,\langle 1\,2\rangle\,\langle 4\,5\rangle\,\langle 5\,6\rangle\,[7|K_{456}|3\rangle[7|K_{56}|4\rangle} \\ &\times (H_3^2(\langle m|P_{12}, m, \langle m|P_{12}, m; 3,3,3; 1, \langle 6|K_{45}K_{123}, 2; P_{12}) \\ &+ \langle 3\,2\rangle\, s_{12} H_3^{1;x}(\langle m|P_{12}, m; m, m, 3, 3; 1, \langle 6|K_{45}K_{123}; 2; P_{12}) \\ &+ \langle 3\,2\rangle\,\langle m\,2\rangle\, s_{12}^2 H_3^{0;xx}(m,m,m,3,3;1,\langle 6|K_{45}K_{123};2;P_{12}))\,. \end{aligned} \tag{5.5.12}$$

Returning to solving D_A^{1b}, we start out with

$$D_A^{1b} = \frac{\langle m\, l_1\rangle^2\,\langle m\, l_2\rangle^2\,\langle 6|K_{7l_1}K_{l_2 3}|4\rangle([l_2|P_{67}|l_1\rangle\,\langle 6\,4\rangle - [l_2|5|4\rangle\,\langle l_1\, 6\rangle)^2}{\langle 4\,5\rangle\,\langle 5\,6\rangle\,\langle 6\,7\rangle\,\langle 7\, l_1\rangle\,[l_2\, 3]\,[3|K_{7l_1 l_2}|6\rangle[l_2|K_{7l_1}|6\rangle\langle l_1|K_{l_2 3}K_{45}|6\rangle\,\langle 1\,2\rangle\,\langle 2\, l_2\rangle\,\langle l_2\, l_1\rangle\,\langle l_1\, 1\rangle}. \tag{5.5.13}$$

The two problematic terms are in the numerator; we have first the term $\langle 6|K_{7l_1}K_{l_2 3}|4\rangle$ which can be simplified somewhat,

$$\begin{aligned} \langle 6|K_{7l_1}K_{l_2 3}|4\rangle =& \langle 6|K_{7l_1}K_{l_1 567}|4\rangle \\ =& \langle 6|K_{7l_1}K_{7l_1}|4\rangle + \langle 6|K_{7l_1}K_{56}|4\rangle\,, \\ =& [7\, l_1]\,\langle l_1\, 7\rangle\,\langle 6\,4\rangle + \langle 6\, l_1\rangle\,[l_1|K_{56}|4\rangle + \langle 6\,7\rangle\,[7|K_{56}|4\rangle\,. \end{aligned} \tag{5.5.14}$$

The second tricky part is the squared term, $([l_2|K_{67}|l_1\rangle\,\langle 6\,4\rangle - [l_2|5|4\rangle\,\langle l_1\, 6\rangle)^2$, which can be written in a more optimal form,

$$\begin{aligned} ([l_2|K_{67}|l_1\rangle\,\langle 6\,4\rangle - [l_2|5|4\rangle\,\langle l_1\, 6\rangle)^2 =& ([l_2|K_{56}|4\rangle\,\langle 6\, l_1\rangle + \langle 6\,4\rangle\,[l_2\, 7]\,\langle 7\, l_1\rangle)^2 \\ =& ([l_2|K_{56}|4\rangle^2\,\langle 6\, l_1\rangle^2 + 2[l_2|K_{56}|4\rangle\,\langle 6\, l_1\rangle\,\langle 6\,4\rangle\,[l_2\, 7]\,\langle 7\, l_1\rangle + \langle 6\,4\rangle^2\,[l_2\, 7]^2\,\langle 7\, l_1\rangle^2)\,. \end{aligned} \tag{5.5.15}$$

The full term is now

$$D_A^{1b} = \langle m\, l_1\rangle^2 \langle m\, l_2\rangle^2 \left([7\, l_1]\, \langle l_1\, 7\rangle\, \langle 6\, 4\rangle + \langle 6\, l_1\rangle\, [l_1|K_{56}|4\rangle + \langle 6\, 7\rangle\, [7|K_{56}|4\rangle\right)$$
$$\times \frac{\left([l_2|K_{56}|4\rangle^2 \langle 6\, l_1\rangle^2 + 2[l_2|K_{56}|4\rangle\, [l_2\, 7]\, \langle 6\, 4\rangle\, \langle 6\, l_1\rangle\, \langle 7\, l_1\rangle + \langle 6\, 4\rangle^2 [l_2\, 7]^2 \langle 7\, l_1\rangle^2\right)}{\langle 4\, 5\rangle\, \langle 5\, 6\rangle\, \langle 6\, 7\rangle\, \langle 7\, l_1\rangle\, [l_2\, 3]\, [3|K_{7l_1l_2}|6\rangle [l_2|K_{7l_1}|6\rangle \langle l_1|K_{l_23}K_{45}|6\rangle\, \langle 1\, 2\rangle\, \langle 2\, l_2\rangle\, \langle l_2\, l_1\rangle\, \langle l_1\, 1\rangle}. \tag{5.5.16}$$

This expands out to give

$$\begin{aligned}
D_A^{1b} =&(- \langle m\, l_1\rangle^2 \langle m\, l_2\rangle\, [7|l_1|7\rangle \langle 4|K_{56}l_2|m\rangle \langle l_1|P_{12}K_{56}|4\rangle\, \langle 6\, l_1\rangle^2 \langle 6\, 4\rangle \\
&+ 2\, \langle m\, l_1\rangle^2 \langle m\, l_2\rangle\, [7|l_1|7\rangle [7|l_2|m\rangle \langle l_1|P_{12}K_{56}|4\rangle\, \langle 6\, l_1\rangle\, \langle 7\, l_1\rangle\, \langle 6\, 4\rangle^2 \\
&+ \langle m\, l_1\rangle^2 \langle m\, l_2\rangle\, [7|l_1|7\rangle [7|l_2|m\rangle [7|P_{12}|l_1\rangle\, \langle 7\, l_1\rangle^2 \langle 6\, 4\rangle^3 \\
&- \langle m\, l_1\rangle^2 \langle m\, l_2\rangle\, \langle 4|K_{56}l_1|6\rangle \langle 4|K_{56}l_2|m\rangle \langle l_1|P_{12}K_{56}|4\rangle\, \langle 6\, l_1\rangle^2 \\
&+ 2\, \langle m\, l_1\rangle^2 \langle m\, l_2\rangle\, \langle 4|K_{56}l_1|6\rangle [7|l_2|m\rangle \langle l_1|P_{12}K_{56}|4\rangle\, \langle 6\, l_1\rangle\, \langle 7\, l_1\rangle\, \langle 6\, 4\rangle \\
&+ \langle m\, l_1\rangle^2 \langle m\, l_2\rangle\, \langle 4|K_{56}l_1|6\rangle [7|l_2|m\rangle [7|P_{12}|l_1\rangle\, \langle 7\, l_1\rangle^2 \langle 6\, 4\rangle^2 \\
&- \langle m\, l_1\rangle^2 \langle m\, l_2\rangle\, \langle 4|K_{56}l_2|m\rangle \langle l_1|P_{12}K_{56}|4\rangle\, \langle 6\, l_1\rangle^2 \langle 6\, 7\rangle\, [7|K_{56}|4\rangle \\
&+ 2\, \langle m\, l_1\rangle^2 \langle m\, l_2\rangle\, [7|l_2|m\rangle \langle l_1|P_{12}K_{56}|4\rangle\, \langle 6\, l_1\rangle\, \langle 7\, l_1\rangle\, \langle 6\, 4\rangle\, \langle 6\, 7\rangle\, [7|K_{56}|4\rangle \\
&+ \langle m\, l_1\rangle^2 \langle m\, l_2\rangle\, [7|l_2|m\rangle [7|P_{12}|l_1\rangle\, \langle 7\, l_1\rangle^2 \langle 6\, 4\rangle^2 \langle 6\, 7\rangle\, [7|K_{56}|4\rangle) \\
&\times \frac{1}{\langle 1\, 2\rangle\, \langle 4\, 5\rangle\, \langle 5\, 6\rangle\, \langle 6\, 7\rangle\, [3|K_{45}|6\rangle\, \langle 1\, l_1\rangle\, \langle 7\, l_1\rangle\, [3|P_{12}|l_1\rangle \langle l_1|P_{12}K_{345}|6\rangle \langle l_1|K_{123}K_{45}|6\rangle\, \langle 2\, l_2\rangle}.
\end{aligned} \tag{5.5.17}$$

This can be evaluated by the tedious, but straightforward, approach of applying equation (5.2.7) to each of the 9 terms in order. Splitting the l_2 dependent string in the first term, we apply equation (5.2.7) to the quadratic part with the prescription

$$A = 7, B = 7, C = \langle 4|K_{56}, D = m, a_i = (m, m, 6, 6, \langle 4|K_{56}P_{12}),$$
$$b_i = (1, 7, [3|P_{12}, \langle 6|K_{345}P_{12}, \langle 6|K_{45}K_{123}), c_j = m, d_j = 2,$$

The linear term is trivial to homogenize using equation (3.2.9) and we thus obtain the term

$$D^{1b}_{A,1} = \frac{\langle 6\,4\rangle}{\langle 1\,2\rangle\,\langle 4\,5\rangle\,\langle 5\,6\rangle\,\langle 6\,7\rangle\,[3|K_{45}|6\rangle}$$
$$\times (H^2_6(7,7,\langle 4|K_{56},m;m,m,6,6,\langle 4|K_{56}P_{12},m;1,7,[3|P_{12},\langle 6|K_{345}P_{12},\langle 6|K_{45}P_{123},2;P_{12})$$
$$+\,\langle m\,2\rangle\, H^{1;x}_6(7,7;\langle 4|K_{56}P_{12},m,m,m,6,6,\langle 4|K_{56}P_{12};1,7,[3|P_{12},\langle 6|K_{345}P_{12},\langle 6|K_{45}K_{123};2;P_{12})$$
$$+\,\langle m\,2\rangle\,\langle 4|K_{56}P_{12}|2\rangle$$
$$\times\, H^{0;xx}_6([7|P_{12},7,m,m,m,6,6,\langle 4|K_{56}P_{12};1,7,[3|P_{12},\langle 6|K_{345}P_{12},\langle 6|K_{45}K_{123};2;P_{12})$$
$$+\langle 4|K_{56}P_{12}|m\rangle H^1_6(7,7;m,m,6,6,\langle 4|K_{56}P_{12},m;1,7,[3|P_{12},\langle 6|K_{345}P_{12},\langle 6|K_{45}K_{123},2;P_{12})$$
$$-\langle 4|K_{56}P_{12}|m\rangle\,\langle m\,2\rangle$$
$$\times\, H^{0;x}_6(7,[7|P_{12},m,m,6,6,\langle 4|K_{56}P_{12};1,7,[3|P_{12},\langle 6|K_{345}P_{12},\langle 6|K_{45}K_{123};2;P_{12}))\,. \tag{5.5.18}$$

With one term solved, terms $D^{1b}_{A,2}$ to $D^{1b}_{A,6}$ are relatively simple, being essentially the same calculation with slightly different choice of parameters. Firstly, we get $D^{1b}_{A,2}$,

$$D^{1b}_{A,2} = \frac{2\,\langle 6\,4\rangle^2}{\langle 1\,2\rangle\,\langle 4\,5\rangle\,\langle 5\,6\rangle\,\langle 6\,7\rangle\,[3|K_{45}|6\rangle}$$
$$\times\,(H^2_6(7,7,7,m;m,m,6,7,\langle 4|K_{56}P_{12},m;1,7,[3|P_{12},\langle 6|K_{345}P_{12},\langle 6|K_{45}P_{123},2;P_{12})$$
$$+\,\langle m\,2\rangle\, H^{1;x}_6(7,7;[7|P_{12},m,m,m,6,7,\langle 4|K_{56}P_{12};1,7,[3|P_{12},\langle 6|K_{345}P_{12},\langle 6|K_{45}K_{123};2;P_{12})$$
$$+\,\langle m\,2\rangle\,[7|P_{12}|2\rangle$$
$$\times\, H^{0;xx}_6([7|P_{12},7,m,m,m,6,7,\langle 4|K_{56}P_{12};1,7,[3|P_{12},\langle 6|K_{345}P_{12},\langle 6|K_{45}K_{123};2;P_{12})$$
$$+\,[7|P_{12}|m\rangle H^1_6(7,7;m,m,6,7,\langle 4|K_{56}P_{12},m;1,7,[3|P_{12},\langle 6|K_{345}P_{12},\langle 6|K_{45}K_{123},2;P_{12})$$
$$-\,[7|P_{12}|m\rangle\langle m\,2\rangle H^{0;x}_6(7,[7|P_{12},m,m,6,7,\langle 4|K_{56}P_{12};1,7,[3|P_{12},\langle 6|K_{345}P_{12},\langle 6|K_{45}K_{123};2;P_{12}))\,, \tag{5.5.19}$$

followed by $D^{1b}_{A,3}$,

$$D^{1b}_{A,3} = \frac{\langle 6\,4\rangle^3}{\langle 1\,2\rangle\,\langle 4\,5\rangle\,\langle 5\,6\rangle\,\langle 6\,7\rangle\,[3|K_{45}|6\rangle}$$
$$\times\,(H^2_6(7,7,7,m;m,m,7,7,[7|P_{12},m;1,7,[3|P_{12},\langle 6|K_{345}P_{12},\langle 6|K_{45}P_{123},2;P_{12})$$
$$+\,\langle m\,2\rangle\, H^{1;x}_6(7,7;[7|P_{12},m,m,m,7,7,[7|P_{12};1,7,[3|P_{12},\langle 6|K_{345}P_{12},\langle 6|K_{45}K_{123};2;P_{12})$$
$$+\,\langle m\,2\rangle\,[7|P_{12}|2\rangle H^{0;xx}_6([7|P_{12},7,m,m,m,7,7,[7|P_{12};1,7,[3|P_{12},\langle 6|K_{345}P_{12},\langle 6|K_{45}K_{123};2;P_{12})$$
$$+\,[7|P_{12}|m\rangle H^1_6(7,7;m,m,7,7,[7|P_{12},m;1,7,[3|P_{12},\langle 6|K_{345}P_{12},\langle 6|K_{45}K_{123},2;P_{12})$$
$$-\,[7|P_{12}|m\rangle\,\langle m\,2\rangle\, H^{0;x}_6(7,[7|P_{12},m,m,7,7,[7|P_{12};1,7,[3|P_{12},\langle 6|K_{345}P_{12},\langle 6|K_{45}K_{123};2;P_{12}))\,. \tag{5.5.20}$$

$D^{1b}_{A,4}$ through $D^{1b}_{A,6}$ are virtually identical, except for the different choice $A = \langle 4|P_{56}$, $B = 6$ for equation (5.2.7),

$$D^{1b}_{A,4} = -\frac{1}{\langle 1\,2\rangle\,\langle 4\,5\rangle\,\langle 5\,6\rangle\,\langle 6\,7\rangle\,[3|K_{45}|6\rangle}$$
$$\times\,(H^2_6(\langle 4|P_{56},6,\langle 4|K_{56},m;m,m,6,6,\langle 4|K_{56}P_{12},m;1,7,[3|P_{12},\langle 6|K_{345}P_{12},\langle 6|K_{45}P_{123},2;P_{12})$$
$$+\,\langle m\,2\rangle\,H^{1;x}_6(\langle 4|P_{56},6;\langle 4|K_{56}P_{12},m,m,m,6,6,\langle 4|K_{56}P_{12};1,7,[3|P_{12},\langle 6|K_{345}P_{12},\langle 6|K_{45}K_{123};2;P_{12})$$
$$+\,\langle m\,2\rangle\,\langle 4|K_{56}P_{12}|2\rangle$$
$$\times\,H^{0;xx}_6(\langle 4|P_{56}P_{12},6,m,m,m,6,6,\langle 4|K_{56}P_{12};1,7,[3|P_{12},\langle 6|K_{345}P_{12},\langle 6|K_{45}K_{123};2;P_{12})$$
$$+\,\langle 4|K_{56}P_{12}|m\rangle H^1_6(\langle 4|P_{56},6;m,m,6,6,\langle 4|K_{56}P_{12},m;1,7,[3|P_{12},\langle 6|K_{345}P_{12},\langle 6|K_{45}K_{123},2;P_{12})$$
$$-\,\langle 4|K_{56}P_{12}|m\rangle\,\langle m\,2\rangle$$
$$\times\,H^{0;x}_6(\langle 4|P_{56}P_{12},6,m,m,6,6,\langle 4|K_{56}P_{12};1,7,[3|P_{12},\langle 6|K_{345}P_{12},\langle 6|K_{45}K_{123};2;P_{12}))\,, \tag{5.5.21}$$

$$D^{1b}_{A,5} = \frac{2\,\langle 6\,4\rangle}{\langle 1\,2\rangle\,\langle 4\,5\rangle\,\langle 5\,6\rangle\,\langle 6\,7\rangle\,[3|K_{45}|6\rangle}$$
$$\times\,(H^2_6(\langle 4|P_{56},6,7,m;m,m,6,7,\langle 4|K_{56}P_{12},m;1,7,[3|P_{12},\langle 6|K_{345}P_{12},\langle 6|K_{45}P_{123},2;P_{12})$$
$$+\,\langle m\,2\rangle\,H^{1;x}_6(\langle 4|P_{56},6;[7|P_{12},m,m,m,6,7,\langle 4|K_{56}P_{12};1,7,[3|P_{12},\langle 6|K_{345}P_{12},\langle 6|K_{45}K_{123};2;P_{12})$$
$$+\,\langle m\,2\rangle\,[7|P_{12}|2\rangle$$
$$\times\,H^{0;xx}_6(\langle 4|P_{56}P_{12},6,m,m,m,6,7,\langle 4|K_{56}P_{12};1,7,[3|P_{12},\langle 6|K_{345}P_{12},\langle 6|K_{45}K_{123};2;P_{12})$$
$$+\,[7|P_{12}|m\rangle H^1_6(\langle 4|P_{56},6;m,m,6,7,\langle 4|K_{56}P_{12},m;1,7,[3|P_{12},\langle 6|K_{345}P_{12},\langle 6|K_{45}K_{123},2;P_{12})$$
$$-\,[7|P_{12}|m\rangle\,\langle m\,2\rangle$$
$$\times\,H^{0;x}_6(\langle 4|P_{56}P_{12},6,m,m,6,7,\langle 4|K_{56}P_{12};1,7,[3|P_{12},\langle 6|K_{345}P_{12},\langle 6|K_{45}K_{123};2;P_{12}))\,, \tag{5.5.22}$$

$$D^{1b}_{A,6} = -\frac{\langle 6\,4\rangle^2}{\langle 1\,2\rangle\,\langle 4\,5\rangle\,\langle 5\,6\rangle\,\langle 6\,7\rangle\,[3|K_{45}|6\rangle}$$
$$\times\,(H^2_6(\langle 4|P_{56},6,7,m;m,m,7,7,[7|P_{12},m;1,7,[3|P_{12},\langle 6|K_{345}P_{12},\langle 6|K_{45}P_{123},2;P_{12})$$
$$+\,\langle m\,2\rangle\,H^{1;x}_6(\langle 4|P_{56},6;[7|P_{12},m,m,m,7,7,[7|P_{12};1,7,[3|P_{12},\langle 6|K_{345}P_{12},\langle 6|K_{45}K_{123};2;P_{12})$$
$$+\,\langle m\,2\rangle\,[7|P_{12}|2\rangle H^{0;xx}_6(\langle 4|P_{56}P_{12},6,m,m,m,7,7,[7|P_{12};1,7,[3|P_{12},\langle 6|K_{345}P_{12},\langle 6|K_{45}K_{123};2;P_{12})$$
$$+\,[7|P_{12}|m\rangle H^1_6(\langle 4|P_{56},6;m,m,7,7,[7|P_{12},m;1,7,[3|P_{12},\langle 6|K_{345}P_{12},\langle 6|K_{45}K_{123},2;P_{12})$$
$$-\,[7|P_{12}|m\rangle\,\langle m\,2\rangle\,H^{0;x}_6(\langle 4|P_{56}P_{12},6,m,m,7,7,[7|P_{12};1,7,[3|P_{12},\langle 6|K_{345}P_{12},\langle 6|K_{45}K_{123};2;P_{12}))\,. \tag{5.5.23}$$

The remaining three terms are linear in l, and thus relatively straightforward,

$$D^{1b}_{A,7} = \frac{\langle 6\,7\rangle\,[7|P_{56}|4\rangle}{\langle 1\,2\rangle\,\langle 4\,5\rangle\,\langle 5\,6\rangle\,\langle 6\,7\rangle\,[3|K_{45}|6\rangle}$$
$$\times\,(H^1_6(\langle 4|P_{56},m;m,m,6,6,\langle 4|P_{56}P_{12},m;1,7,[3|P_{12},\langle 6|P_{345}P_{12},\langle 6|P_{45}P_{123},2;P_{12})$$
$$+\,\langle m\,2\rangle\,H^{0;x}_6(\langle 4|P_{56}P_{12},m,m,m,6,6,\langle 4|P_{56}P_{12};1,7,[3|P_{12},\langle 6|P_{345}P_{12},\langle 6|P_{45}P_{123};2;P_{12})$$
$$+\,\langle m|P_{12}P_{56}|4\rangle H^0_6(m,m,m,6,6,\langle 4|P_{56}P_{12};1,7,[3|P_{12},\langle 6|P_{345}P_{12},\langle 6|P_{45}P_{123},2;P_{12}))\,, \tag{5.5.24}$$

$$
\begin{aligned}
D^{1b}_{A,8} =& \frac{2\,\langle 6\,4\rangle\,\langle 6\,7\rangle\,[7|P_{56}|4\rangle}{\langle 1\,2\rangle\,\langle 4\,5\rangle\,\langle 5\,6\rangle\,\langle 6\,7\rangle\,[3|K_{45}|6\rangle} \\
&\times \big(H^1_6(7,m;m,m,6,7,\langle 4|P_{56}P_{12},m;1,7,[3|P_{12},\langle 6|P_{345}P_{12},\langle 6|P_{45}P_{123},2;P_{12}) \\
&- \langle m\,2\rangle\,H^{0;x}_6([7|P_{12},m,m,m,6,7,\langle 4|P_{56}P_{12};1,7,[3|P_{12},\langle 6|P_{345}P_{12},\langle 6|P_{45}P_{123};2;P_{12}) \\
&+ [7|P_{12}|m\rangle H^0_6(m,m,m,6,7,\langle 4|P_{56}P_{12};1,7,[3|P_{12},\langle 6|P_{345}P_{12},\langle 6|P_{45}P_{123},2;P_{12})\big)\,,
\end{aligned}
\tag{5.5.25}
$$

$$
\begin{aligned}
D^{1b}_{A,9} =& -\frac{\langle 6\,4\rangle^2\,\langle 6\,7\rangle\,[7|P_{56}|4\rangle}{\langle 1\,2\rangle\,\langle 4\,5\rangle\,\langle 5\,6\rangle\,\langle 6\,7\rangle\,[3|K_{45}|6\rangle} \\
&\times \big(H^1_6(7,m;m,m,7,7,[7|P_{12},m;1,7,[3|P_{12},\langle 6|P_{345}P_{12},\langle 6|P_{45}P_{123},2;P_{12}) \\
&- \langle m\,2\rangle\,H^{0;x}_6([7|P_{12},m,m,m,7,7,[7|P_{12};1,7,[3|P_{12},\langle 6|P_{345}P_{12},\langle 6|P_{45}P_{123};2;P_{12}) \\
&+ [7|P_{12}|m\rangle H^0_6(m,m,m,7,7,[7|P_{12};1,7,[3|P_{12},\langle 6|P_{345}P_{12},\langle 6|P_{45}P_{123},2;P_{12})\big)\,.
\end{aligned}
\tag{5.5.26}
$$

With the full D_A cut calculated, we can solve the remaining bubble coefficients of the A amplitude,

$$
d^A_3 = -D_A(4,3,2,1,7,6,5;3,P_{34}),\;\; d^A_7 = D_A(7,1,2,3,4,5,6;1,P_{71}), \tag{5.5.27}
$$

As with the six point, one can verify as a consistency check that the bubble coefficients satisfy the IR constraint,

$$
\sum d_i + \sum c_i = \frac{1}{3}A^{tree}(1^-,2^-,3^-,4^+,5^+,6^+,7^+)\,. \tag{5.5.28}
$$

Chapter 6

Conclusions

6.1 Summary

In Chapter 1 a case was presented for why efficient methods for performing accurate calculations of NLO amplitudes are important for LHC phenomenology. The problems with the conventional Feynman diagram approach when applied to NLO amplitudes with many external particles were outlined, and an overview was given of the contemporary approaches to overcoming those obstacles and computing NLO processes from the Les Houches wishlist.

Although there are a number of approaches to exploiting the cut-constructibility of loop amplitudes in order to compute processes with many external particles, each comes with drawbacks. The OPP integrand reduction method and BlackHat automation of the Bootstrap approach are both aimed towards numerical implementation, and indeed both have reached a high degree of maturity in doing so as evidenced by their application to non-trivial processes from the Les Houches wishlist. The techniques of D-dimensional Unitarity provide powerful advantages over 4-dimensional methods, most notably the ability to compute the parts of the amplitude which are not cut-constructible in 4-dimensions, however they are complicated by the difficulty of extending the tree amplitudes to D-dimensions, and the method loses the essential simplicity of quadruple cuts in 4-dimensions, while gaining the necessity that one also consider pentuple cuts. As such this method has also been found to be best suited to an automated, numerical approach, in the form of the Rocket package.

While numerical calculations are entirely adequate, indeed preferable, for providing predictions for phenomenological purposes, there is nonetheless value in being able to produce a closed, analytic expression for the coefficients of a loop amplitude. A closed analytic expression enables the structure of the amplitude to be more directly examined, and in particular knowledge of the analytic structure of the amplitude is essential in order to use it as input in recursion. Since recursion is so fundamental to calculation of amplitudes in modern theories, with both Unitarity and on-shell recursion relying upon constructing amplitudes from known, simpler amplitudes, this represents a serious limitation upon purely numerical techniques.

After an overview of some relevant techniques and formalism and an explanation of the Unitarity method itself in Chapter 2, in Chapter 3 the Canonical Basis approach was introduced as a Unitarity method geared towards producing analytic, rational expressions for loop amplitudes in Yang-Mills theories. The key to the method is to construct some basis of possible loop-momentum dependent structures which can appear in the integrand of a Unitarity cut, and to systematically solve each individual canonical form. The method used to perform the cut integration and extract the contribution to the bubble coefficient does not matter; for the canonical basis constructed here, only two canonical forms were integrated directly; all other canonical forms were found to be reducible to these two by reduction using a number of identities. The result was a basis by which one could easily extract the bubble contribution to a large class of integrand structures simply by inserting the previously solved, analytic, fully rational expression for the canonical form.

In Chapter 4 the constructed canonical basis was applied to a previously unknown loop amplitude, the case of a NMHV seven-gluon amplitude with an $\mathcal{N} = 1$ chiral supermultiplet circulating in the loop. The relatively small number of unique double cut configurations was identified, and each case was reduced to a sum of canonical forms evaluated in Chapter 3, multiplying some non-l-dependent coefficient. In this manner the complete set of bubble coefficients for the 7-gluon NMHV amplitude with an $\mathcal{N} = 1$ Chiral multiplet loop was constructed. A general form was then given for the triangle coefficients using the previously-constructed canonical basis for the triple cuts [79], exploiting the extremely restricted set of valid triple cuts possible at 7-point; finally, the box coefficients were found using quadruple cuts. Since supersymmetric loops contain no rational terms and are thus cut-constructible in 4 dimensions, this thus represents a complete analytic expression for the 7-gluon NMHV $\mathcal{N} = 1$ Chiral multiplet loop contribution.

In Chapter 5 the method was extended to consider the extraction of the coefficients of the cut-constructible parts of non-SUSY amplitudes, specifically the case of the 6-gluon amplitude with a complex scalar circulating in the loop. This introduces significant complications owing to the presence of terms in the integrand which are quadratic or linear in the loop momentum in addition to terms of $\mathcal{O}(l^0)$. Despite the added difficulty of such canonical forms, a basis for the relevant cases was constructed in Chapter 3, and in Chapter 5 it was applied to solve the bubble coefficients of the 6-gluon amplitude. Since this is a previously calculated amplitude [82], this calculation allowed a consistency check on the amplitude and by extension the canonical basis itself beyond those imposed by the structure of the amplitude or the IR behaviour of the $\frac{1}{\epsilon}$ pole, since the bubble coefficients could be explicitly numerically compared to the previous result for the $\frac{1}{\epsilon}$ contribution at the kinematic point specified in that paper.

The triangle and box coefficients were briefly discussed, and the method was then applied to the split-helicity case of the 7-gluon NMHV scalar loop, in order to illustrate the ease by which the 6-point results might be extended to extract the previously-unknown cut-constructible parts of the 7-gluon NMHV amplitude.

6.2 Outlook

There are a number of ways in which the Canonical Basis approach presented here could be extended or improved. In Chapter 3 a canonical basis of double cut integrals was constructed, sufficient to compute all double cuts for all-gluon one-loop amplitudes up to seven point with an $\mathcal{N}=1$ chiral multiplet or a massless complex scalar circulating in the loop. However, although the canonical forms provide a correct analytic solution to their respective cut integrals as indicated by the various tests that were performed upon both the canonical forms themselves and upon the integral coefficients they were used to solve, the canonical forms themselves are not unique solutions to the cut integrals. In particular it is likely they are not the most optimal forms in which the solution to the integrals could be expressed. This is particularly true of the canonical forms for cut integrals with linear or quadratic l dependence, due to the large numbers of independent parameters present in such canonical forms and thus the complicated expressions which arise as a result.

It is often the case that structures appearing in loop integral coefficients are interrelated via the Schouten identity. Schouten identities appear abundantly in coefficients of loop amplitudes with many external particles and often allow a pair or series of terms to be rewritten as a single term. This property is particularly true of the linear and quadratic G-function canonical forms presented in Chapter 3, where it is clear one could improve upon the existing canonical forms by contracting terms together using the Schouten identity. For example one can easily simplify the canonical form $G_1^{0;x}$,

$$\begin{aligned}
&G_1^{0;x}(A,B,C,D,E,Q,P) = \frac{P^2\,[E\,A]\,[E|P|B\rangle\,\langle D\,E\rangle\,\langle C\,E\rangle}{[E|P|E\rangle^2\langle E|QP|E\rangle} \\
&-\frac{P^2\,[A\,E]\,\langle E\,B\rangle\,(\langle D|[Q,P]|E\rangle\,\langle C\,E\rangle+\langle C|[Q,P]|E\rangle\,\langle D\,E\rangle)}{2[E|P|E\rangle\langle E|QP|E\rangle^2} \\
&+\frac{[A|P[P,Q]|B\rangle(\langle D|[Q,P]|E\rangle\langle C|[Q,P]|E\rangle+\Delta_3\,\langle D\,E\rangle\,\langle C\,E\rangle)}{4\Delta_3\langle E|QP|E\rangle^2} \\
&+\frac{[A|P|B\rangle(\langle C|[Q,P]|E\rangle\,\langle D\,E\rangle+\langle D|[Q,P]|E\rangle\,\langle C\,E\rangle)}{4\langle E|QP|E\rangle^2}\,.
\end{aligned} \tag{6.2.1}$$

Another area in which the existing canonical basis could be improved upon is in the consideration of spurious poles. These are poles which appear in a cut at a particular point in momentum space, but which do not correspond to a physical factorization of the amplitude. Since the pole is unphysical, any appearance in a term in the cut must be cancelled by the same pole appearing in another term in the amplitude. This cancellation provides a useful check upon evaluated amplitudes.

One type of spurious pole are Gram determinants, where two non-null momenta P and Q appearing in a double cut appear in the structure $(P.Q)^2 - P^2Q^2$. For an appropriate choice of the momenta this factor appearing in a denominator can become singular when the amplitude itself is not singular. A particular type of Gram determinant dependence, namely dependence upon non-integer powers of the Gram determinants, has been systematically eliminated from all canonical forms appearing in Chapter 3, since it introduces an unphysical irrational part into the necessarily rational integral coefficients. However, other types of spurious singularities, such as integer powers of Gram determinants, and coplanar singularities, remain present in the final integral coefficients. It would be interesting to investigate whether one could systematically eliminate such necessarily unphysical terms from the canonical forms themselves.

The canonical basis approach is particularly amenable to such progressive optimisation. This is due to the inherent modularity of the method; the cut integrands in Chapters 4 and 5 are reduced to expressions composed purely of canonical forms, without reference being made to the particular solution those canonical forms have. It is thus a simple matter to substitute any given canonical form with an improved expression, gaining a significant simplification in the final expression of the amplitude without necessitating any new derivation of the individual cuts.

Another key advantage of the canonical basis approach is the ease with which it can be repeatedly applied to any cut integral with a structure appearing in the canonical basis. This was seen in Chapters 4 and 5; although the derivation of all necessary $\mathcal{O}(l^0)$, $\mathcal{O}(l^1)$ and $\mathcal{O}(l^2)$ canonical forms in Chapter 3 took considerable work, once the basis was solved it was relatively simple to quickly solve many cut integrals in the 7-gluon $\mathcal{N} = 1$ loop and the 6-gluon complex scalar loop by identifying large numbers of cuts with a relatively small number of canonical forms. It is likely that this inherent reusability of the canonical basis could be applied to solve some or all of the cuts of other amplitudes. An obvious extension would be the 7-gluon scalar loop; in Chapter 5 it was demonstrated that with all elements of the canonical basis required for the 6-gluon case in place, one can straightforwardly evaluate the bubble coefficients of the simplest helicity configuration of the 7-gluon case, the $(-,-,-,+,+,+,+)$ configuration, using the same canonical forms, and indeed in some cases simple extensions of the same cuts found at 6-point. It would be straightforward to repeat this process to solve the bubble coefficients of the remaining configurations.

The particular canonical basis presented here is specific to the 4-dimensional massless case, in which one can apply the power of the Spinor Helicity formalism. Two of the most relevant possible extensions of the method would result in the formalism no longer being valid, namely a consideration of amplitudes containing massive particles, and an extension to considering Unitarity cuts in D-dimensions, in order to allow the rational terms in the amplitude to be evaluated by Unitarity. In these cases obviously the particular canonical basis presented in Chapter 3 would no longer be applicable. It would be interesting however to consider whether the general canonical basis approach might still be worthwhile, namely of trying to identify all unique l-dependent structures which could arise from the cut integrand, and solve them in general in order to obtain significant time savings in avoiding evaluating separately large numbers of cuts with the same l dependence. An obstacle to such an approach, however, may be present if the utility of the Spinor Helicity formalism is the ultimate reason why so many cuts in the examples considered in this thesis can be solved with such a relatively small basis. The fact that the Spinor Helicity formalism both severely restricts the number of possible l-dependent structures which can appear in the integrand, and allows for large classes of canonical forms to be written in terms of simpler forms (such as the way in which a canonical form of the type G_n^m can be trivially written as a sum of G_1^m functions), may hint that this may be the case, and that a massive or D-dimensional canonical basis might need to be substantially larger, thus limiting its attractiveness.

6.3 Limitations

Although the canonical basis approach presented in this thesis has many desirable features, it nonetheless also possesses some important limitations. Firstly and perhaps most importantly, since it is fundamentally a 4-dimensional Unitarity technique, it shares the drawback of all such approaches that one cannot obtain from 4-dimensional Unitarity cuts those parts of the amplitude which do not contain logarithmic parts in 4-dimensions. As discussed in chapter 4, for supersymmetric theories this is not a problem; in such theories the cancellations between supermultiplet particle types cause any contributions below the level of scalar bubble integrals to cancel; as such the entire amplitude is cut-constructible. This property does not hold true for non-supersymmetric loop amplitudes, in which Passarino-Veltman reduction necessarily gives rise to terms in which all loop propagators have been cancelled, eliminating the loop integral and leaving only a rational term. A 4-dimensional Unitarity computation of such an amplitude can therefore only ever give a partial solution, and must be combined with some other method being used to compute the rational pieces in order to compute the full loop amplitude.

Another limitation is that the method presented here has thus far only been used to compute amplitudes analytically. While analytical closed expressions for amplitudes in general have a degree of utility, particularly due to their utility in being used as an input for further calculations, there is a trend towards implementing Unitarity techniques in the form of an automated numerical package, to facilitate their use in phenomenological calculations such as those contained in the Les Houches wishlist. Given the existence of a number of such well-developed automated packages [50, 29, 47], the most sensible way to automate the canonical basis would likely be to modify one of these packages to include it.

The difficulty lies in the stage of constructing the canonical basis itself. As demonstrated in Chapter 3, this is a difficult problem to devise a general approach to, as it is often best to select the method by which to compute the canonical form depending on the specific form itself. Similarly optimising the canonical form once a working expression has been found is a difficult process to automate, given that doing so is often a case of manually identifying Schouten identity simplifications or cancellations between spurious poles. As such the most practical way to automate the procedure would likely be in the stage of reducing individual cuts to canonical forms and then inserting the previously-constructed canonical forms.

Appendix A

Integral Basis

As discussed in Chapter 2, an arbitrary n-point loop amplitudes in massless QCD can be represented as a sum of rational coefficients multiplying standard, known, scalar bubble, triangle and box loop integrals, plus some pure rational part R,

$$I(n) = \sum_i \mathcal{D}_i I_i^{(4)} + \sum_j \mathcal{C}_j I_j^{(3)} + \sum_k \mathcal{B}_k I_k^{(2)} + \mathcal{R}\,. \tag{A.1}$$

The form of these scalar basis integrals has been known for some time [85]. The usual convention is to group the integrals themselves into different classes depending on how many corners of the integral have "massive" external momenta, i.e. have more than one external particle, thus having a total momentum for the corner which is off-shell. For the box integrals, one obtains one-mass, two-mass, three-mass and four-mass boxes, of which only the first three appear for $n \leq 7$. It is conventional to subdivide the two-mass case into two-mass "hard" and "easy" depending upon whether or not the massive corners are adjacent,

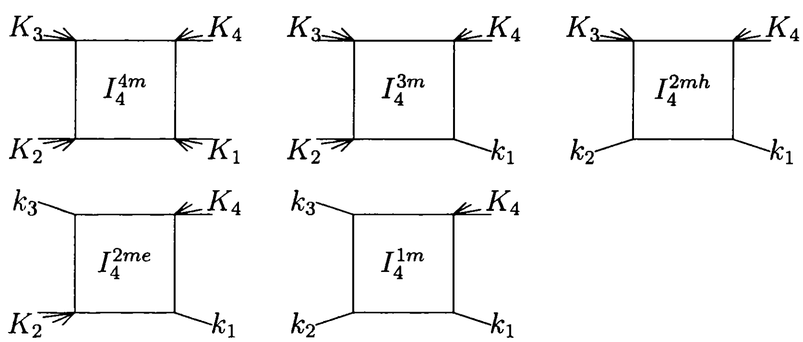


Figure A.1: The different types of scalar box

The scalar integral itself is given by

$$I_4 = -i(4\pi)^{2-\epsilon} \int \frac{d^{4-2\epsilon}l}{(2\pi)^{4-2\epsilon}} \frac{1}{l^2(l-K_1)^2(l-K_1-K_2)^2(l+K_4)^2}\,. \tag{A.2}$$

The solutions to the four mass boxes are given in terms of the non-zero K_i^2 and the invariants S and T, where

$$S \equiv (K_1+K_2)^2,\;\; T \equiv (K_2+K_3)^2\,. \tag{A.3}$$

The convention is to define the scalar box function F where

$$
\begin{aligned}
F(K_1,K_2,K_3,K_4) =& -\frac{2\sqrt{\det S}}{r_\Gamma} I_4\,,\\
r_\Gamma =& \frac{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\,,\\
S_{ij} =& -\frac{1}{2}(K_i+\cdots+K_{j-1})^2,\quad S_{ii}=0\,.
\end{aligned}
\tag{A.4}
$$

In this case we obtain

$$
\begin{aligned}
I_4^{1m} &= -2r_\Gamma\frac{F^{1m}}{ST},\quad I_4^{2me} = -2r_\Gamma\frac{F^{2me}}{ST-K_2^2K_4^2},\\
I_4^{2mh} &= -2r_\Gamma\frac{F^{2mh}}{ST},\quad I_4^{3m} = -2r_\Gamma\frac{F^{3m}}{ST-K_2^2K_4^2},
\end{aligned}
\tag{A.5}
$$

With these conventions the box functions are given to $\mathcal{O}(\epsilon^0)$ by [85]

$$
\begin{aligned}
F^{1m} =& -\frac{1}{\epsilon^2}[(-S)^{-\epsilon}+(-T)^{-\epsilon}-(-K_4^2)^{-\epsilon}]\\
&+\mathrm{Li}_2(1-\frac{K_4^2}{S})+\mathrm{Li}_2(1-\frac{K_4^2}{T})+\frac{1}{2}\ln^2(\frac{S}{T})+\frac{\pi^2}{6}\,,\\
\\
F^{2me} =& -\frac{1}{\epsilon^2}[(-S)^{-\epsilon}+(-T)^{-\epsilon}-(-K_2^2)^{-\epsilon}-(-K_4^2)^{-\epsilon}]\\
&+\mathrm{Li}_2(1-\frac{K_2^2}{S})+\mathrm{Li}_2(1-\frac{K_2^2}{T})+\mathrm{Li}_2(1-\frac{K_4^2}{S})\\
&+\mathrm{Li}_2(1-\frac{K_4^2}{T})-\mathrm{Li}_2(1-\frac{K_2^2K_4^2}{ST})+\frac{1}{2}\ln^2(\frac{S}{T})\,,\\
\\
F^{2mh} =& -\frac{1}{\epsilon^2}[(-S)^{-\epsilon}+(-T)^{-\epsilon}-(-K_3^2)^{-\epsilon}-(-K_4^2)^{-\epsilon}]\\
&-\frac{1}{2\epsilon^2}\frac{(-K_3^2)^{-\epsilon}(-K_4^2)^{-\epsilon}}{(-S)^{-\epsilon}}+\frac{1}{2}\ln^2(\frac{S}{T})\\
&+\mathrm{Li}_2(1-\frac{K_3^2}{T})+\mathrm{Li}_2(1-\frac{K_4^2}{T})\,,\\
\\
F^{3m} =& -\frac{1}{\epsilon^2}[(-S)^{-\epsilon}+(-T)^{-\epsilon}-(-K_2^2)^{-\epsilon}-(-K_3^2)^{-\epsilon}-(-K_4^2)^{-\epsilon}]\\
&-\frac{1}{2\epsilon^2}\frac{(-K_2^2)^{-\epsilon}(-K_3^2)^{-\epsilon}}{(-T)^{-\epsilon}}-\frac{1}{2\epsilon^2}\frac{(-K_3^2)^{-\epsilon}(-K_4^2)^{-\epsilon}}{(-S)^{-\epsilon}}+\frac{1}{2}\ln^2(\frac{S}{T})\\
&+\mathrm{Li}_2(1-\frac{K_2^2}{S})+\mathrm{Li}_2(1-\frac{K_4^2}{T})+\mathrm{Li}_2(1-\frac{K_2^2K_4^2}{ST})\,.
\end{aligned}
\tag{A.6}
$$

A similar subdivision and labeling applies to the triangle integrals, which may be one-mass, two-mass or three-mass. The integrals themselves are given by

$$
\begin{aligned}
I_3^{1m} =& \frac{r_\Gamma}{\epsilon^2}(-K_1^2)^{-1-\epsilon}\,,\\
I_3^{2m} =& \frac{r_\Gamma}{\epsilon^2}\frac{(-K_1^2)^{-\epsilon}-(-K_2^2)^{-\epsilon}}{(-K_1^2)-(-K_2^2)}\,,\\
I_3^{3m} =& \frac{i}{\sqrt{\Delta_3}}\sum_{j=1}^{3}[\mathrm{Li}_2(-\frac{1+i\delta_j}{1-i\delta_j})-\mathrm{Li}_2(-\frac{1-i\delta_j}{1+i\delta_j})]+\mathcal{O}(\epsilon)\,,
\end{aligned}
\tag{A.7}
$$

where the δ_j are defined as

$$
\begin{aligned}
\delta_1 =& \frac{K_1^2-K_2^2-K_3^2}{\sqrt{\Delta_3}}\,,\\
\delta_2 =& \frac{-K_1^2+K_2^2-K_3^2}{\sqrt{\Delta_3}}\,,\\
\delta_3 =& \frac{-K_1^2-K_2^2+K_3^2}{\sqrt{\Delta_3}}\,.
\end{aligned}
\tag{A.8}
$$

An important observation was made by Britto, Buchbinder, Cachazo and Feng [37] regarding the pole structure of the 1- and 2-mass triangle integrals. One can note that both functions contain only $(-K)^{-\epsilon}$ functions multiplying $\frac{1}{\epsilon^2}$ poles. As discussed in chapters 4 and 5, the IR behaviour of loop amplitudes have a very specific structure in Yang-Mills theories, specifically [84]

$$
\begin{aligned}
A_{IR}^{\mathcal{N}=1\text{ chiral}} =& \frac{c_\Gamma}{\epsilon}A^{\text{tree}},\\
A_{IR}^{[0]} =& \frac{c_\Gamma}{3\epsilon}A^{\text{tree}},
\end{aligned}
\tag{A.9}
$$

which implies that any $\frac{1}{\epsilon^2}$ singularities must cancel from the full amplitude. Specifically, since the only other source of $\frac{1}{\epsilon^2}$ poles are the box functions, it is necessary that the coefficients of the 1- and 2-mass triangles must cancel against them, and thus one can include their contribution without explicitly calculating them by absorbing them into the definition of the box functions, defining a set of truncated box functions $\mathcal{F}$

$$
\begin{aligned}
\mathcal{F}^{1m} &= F^{1m}+\frac{1}{\epsilon^2}\left(\frac{\mu^2}{-S}\right)^\epsilon+\frac{1}{\epsilon^2}\left(\frac{\mu^2}{-T}\right)^\epsilon-\frac{1}{\epsilon^2}\left(\frac{\mu^2}{-K_4^2}\right)^\epsilon,\\
\mathcal{F}^{2me} &= F^{2me}+\frac{1}{\epsilon^2}\left(\frac{\mu^2}{-S}\right)^\epsilon+\frac{1}{\epsilon^2}\left(\frac{\mu^2}{-T}\right)^\epsilon-\frac{1}{\epsilon^2}\left(\frac{\mu^2}{-K_2^2}\right)^\epsilon-\frac{1}{\epsilon^2}\left(\frac{\mu^2}{-K_4^2}\right)^\epsilon,\\
\mathcal{F}^{2mh} &= F^{2mh}+\frac{1}{2\epsilon^2}\left(\frac{\mu^2}{-S}\right)^\epsilon+\frac{1}{\epsilon^2}\left(\frac{\mu^2}{-T}\right)^\epsilon-\frac{1}{2\epsilon^2}\left(\frac{\mu^2}{-K_3^2}\right)^\epsilon-\frac{1}{2\epsilon^2}\left(\frac{\mu^2}{-K_4^2}\right)^\epsilon,\\
\mathcal{F}^{3m} &= F^{3m}+\frac{1}{2\epsilon^2}\left(\frac{\mu^2}{-S}\right)^\epsilon+\frac{1}{\epsilon^2}\left(\frac{\mu^2}{-T}\right)^\epsilon-\frac{1}{2\epsilon^2}\left(\frac{\mu^2}{-K_2^2}\right)^\epsilon-\frac{1}{2\epsilon^2}\left(\frac{\mu^2}{-K_4^2}\right)^\epsilon.
\end{aligned}
\tag{A.10}
$$

The full loop amplitude can now be expressed in the form

$$A^{\text{1-loop}} = \sum a_i \mathcal{F}^i + \sum b_j^{3m} I_3^{3m,j} + \sum c_k I_2^k + R\,, \tag{A.11}$$

and the coefficients of the I_3^{1m} and I_3^{2m} do not need to be calculated. In this expression the coefficients of the $\mathcal{F}^i$, $a_{\mathcal{F}}$ are related to the a_I by

$$\begin{aligned} a_{\mathcal{F}}^{1m} &= -\frac{2}{ST} a_I^{1m}, \;\; a_{\mathcal{F}}^{2me} = -\frac{2}{ST - K_2^2 K_4^2} a_I^{2me}, \\ a_{\mathcal{F}}^{2mh} &= -\frac{2}{ST} a_I^{2mh}, \;\; a_{\mathcal{F}}^{3m} = -\frac{2}{ST - K_2^2 K_4^2} a_I^{3m}. \end{aligned} \tag{A.12}$$

The bubble integrals are not subdivided by massive corners since on-shell bubble integrals do not contribute in massless theories. The relevant bubble integral is thus given by

$$I_2(P) = \frac{1}{\epsilon} + 2 - \ln(-P^2)\,. \tag{A.13}$$

Appendix B

Seven-point Tree

We have

$$A: A(s_1,\bar{s}_2,3^-,4^-,5^+,6^+,7^+) = T^A_{1a} + T^A_{1b} + T^A_2 + T^A_3\,, \tag{B.1}$$

with

$$\begin{aligned}
T^A_{1a} =& \frac{[5|P_{345}|2\rangle[5|P_{345}|1\rangle^2}{\langle 6\,7\rangle\,\langle 7\,1\rangle\,\langle 1\,2\rangle\,[3\,4]\,[4\,5]\,[3|P_{345}|6\rangle t_{345}} \times \left(\frac{[5|P_{345}|1\rangle}{[5|P_{345}|2\rangle}\right)^{2h},\\
T^A_{1b} =& \frac{\langle 6|P_{71}P_{23}|4\rangle([2|P_{67}|1\rangle\,\langle 6\,4\rangle - [2|5|4\rangle\,\langle 1\,6\rangle)^2}{\langle 4\,5\rangle\,\langle 5\,6\rangle\,\langle 6\,7\rangle\,\langle 7\,1\rangle\,[2\,3]\,[3|P_{712}|6\rangle[2|P_{71}|6\rangle\langle 1|P_{23}P_{45}|6\rangle} \times \left(\frac{([2|P_{67}|1\rangle\,\langle 6\,4\rangle - [2|5|4\rangle\,\langle 1\,6\rangle)}{\langle 6|P_{71}P_{23}|4\rangle}\right)^{2h},\\
T^A_2 =& \frac{\langle 3\,4\rangle^4}{\langle 3\,5\rangle^4} \times T^B_2\,,\\
T^A_3 =& \frac{[7|P_{456}|4\rangle^4}{[7|P_{456}|5\rangle^4} \times T^B_3\,.
\end{aligned} \tag{B.2}$$

$$B: A(s_1,\bar{s}_2,3^-,4^+,5^-,6^+,7^+) = T^B_{1a} + T^B_{1b} + T^B_{1c} + T^B_2 + T^B_3 + T^B_4\,, \tag{B.3}$$

with

$$\begin{aligned}
T^B_{1a} =& \frac{[4|P_{234}|5\rangle^2\,[2\,4]^2\,\langle 5\,1\rangle^2}{\langle 5\,6\rangle\,\langle 6\,7\rangle\,\langle 7\,1\rangle\,[2\,3]\,[3\,4]\,t_{234}[2|P_{234}|5\rangle[4|P_{234}|1\rangle} \times \left(\frac{[2\,4]\,\langle 5\,1\rangle}{[4|P_{234}|5\rangle}\right)^{2h},\\
T^B_{1b} =& \frac{-\langle 2\,3\rangle\,\langle 1\,3\rangle^2\,[4|P_{67}|1\rangle^4}{\langle 6\,7\rangle\,\langle 7\,1\rangle\,\langle 1\,2\rangle\,[4\,5]\,[5|P_{67}|1\rangle[4|P_{23}|1\rangle\langle 1|P_{67}P_{45}|3\rangle\langle 6|P_{45}P_{23}|1\rangle} \times \left(\frac{\langle 1\,3\rangle}{\langle 2\,3\rangle}\right)^{2h},\\
T^B_{1c} =& \frac{t_{671}\,[2|P_{671}|1\rangle^2\,\langle 3\,5\rangle^4}{\langle 6\,7\rangle\,\langle 7\,1\rangle\,\langle 3\,4\rangle\,\langle 4\,5\rangle\,[2|P_{71}|6\rangle[2|P_{345}|5\rangle\langle 1|P_{67}P_{45}|3\rangle t_{345}} \left(\frac{-[2|P_{671}|1\rangle}{t_{671}}\right)^{2h},\\
T^B_2 =& \frac{-[2\,7]^2\,[1\,7]\,\langle 3\,5\rangle^4}{[1\,2]\,\langle 3\,4\rangle\,\langle 4\,5\rangle\,\langle 5\,6\rangle\,[2|P_{712}|6\rangle[7|P_{712}|3\rangle t_{712}} \times \left(-\frac{[2\,7]}{[1\,7]}\right)^{2h},\\
T^B_3 =& \frac{\langle 1\,3\rangle^2\,\langle 2\,3\rangle^2\,[7|P_{123}|5\rangle^4}{\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 4\,5\rangle\,\langle 5\,6\rangle\,[7|P_{123}|3\rangle[7|P_{123}|4\rangle\langle 1|P_{23}P_{45}|6\rangle t_{123}t_{456}} \times \left(\frac{\langle 1\,3\rangle}{\langle 2\,3\rangle}\right)^{2h},\\
T^B_4 =& \frac{-\langle 1\,3\rangle^2\,\langle 2\,3\rangle^2\,\langle 5\,6\rangle^2\,[6\,7]^3}{\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 3\,4\rangle\,\langle 5\,6\rangle\,[7|P_{56}|4\rangle[5|P_{567}|1\rangle t_{567}s_{56}} \times \left(\frac{\langle 1\,3\rangle}{\langle 2\,3\rangle}\right)^{2h}.
\end{aligned} \tag{B.4}$$

$$C: A(s_1,\bar{s}_2,3^-,4^+,5^+,6^-,7^+) = T^C_{1a} + T^C_{1b} + T^C_{1c} + T^C_2 + T^C_3 + T^C_4\,, \tag{B.5}$$

with

$$
\begin{aligned}
T_{1a}^{C} =& \frac{[2\,4]^2\,[4|P_{234}|6\rangle^2\,\langle 1\,6\rangle^2}{[2\,3]\,[3\,4]\,\langle 5\,6\rangle\,\langle 6\,7\rangle\,\langle 7\,1\rangle\,[4|P_{234}|1\rangle[2|P_{234}|5\rangle t_{234}} \times \left(\frac{-[2\,4]\,\langle 1\,6\rangle}{[4|P_{234}|6\rangle}\right)^{2h}, \\
T_{1b}^{C} =& -\frac{\langle 1\,3\rangle^2\,\langle 2\,3\rangle\,[4\,5]^3\,\langle 1\,6\rangle^4}{\langle 6\,7\rangle\,\langle 7\,1\rangle\,\langle 1\,2\rangle\,[4|P_{234}|1\rangle[5|P_{234}|1\rangle\langle 1|P_{67}P_{45}|3\rangle\langle 1|P_{23}P_{45}|6\rangle}\left(\frac{\langle 1\,3\rangle}{\langle 2\,3\rangle}\right)^{2h}, \\
T_{1c}^{C} =& \frac{[2|P_{345}|3\rangle^2\langle 6|P_{71}P_{345}|3\rangle^2\,\langle 1\,6\rangle^2}{\langle 6\,7\rangle\,\langle 7\,1\rangle\,\langle 3\,4\rangle\,\langle 4\,5\rangle\,[2|P_{71}|6\rangle[2|P_{345}|5\rangle\langle 1|P_{671}P_{345}|3\rangle t_{345}t_{671}} \times \left(\frac{[2|P_{345}|3\rangle\,\langle 1\,6\rangle}{\langle 6|P_{671}P_{345}|3\rangle}\right)^{2h}, \\
T_{2}^{C} =& \frac{[1\,7]^2\,[2\,7]^2\,\langle 3\,6\rangle^4}{[7\,1]\,[1\,2]\,\langle 3\,4\rangle\,\langle 4\,5\rangle\,\langle 5\,6\rangle\,[7|P_{712}|3\rangle[2|P_{712}|6\rangle t_{712}}\left(-\frac{[2\,7]}{[1\,7]}\right)^{2h}, \\
T_{3}^{C} =& \frac{\langle 1\,3\rangle^2\,\langle 2\,3\rangle^2\,[7|P_{456}|6\rangle^4}{\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 4\,5\rangle\,\langle 5\,6\rangle\,\langle 1|P_{123}P_{456}|6\rangle[7|P_{456}|4\rangle[7|P_{456}|3\rangle t_{123}t_{456}}\left(\frac{\langle 1\,3\rangle}{\langle 2\,3\rangle}\right)^{2h}, \\
T_{4}^{C} =& \frac{[7\,5]^4\,\langle 1\,3\rangle^2\,\langle 2\,3\rangle^2}{\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 3\,4\rangle\,[5\,6]\,[6\,7]\,[5|P_{567}|1\rangle[7|P_{567}|4\rangle t_{567}}\left(\frac{\langle 3\,1\rangle}{\langle 3\,2\rangle}\right)^{2h}.
\end{aligned}
\tag{B.6}
$$

$$
D : A_7(s_1, \bar{s}_2, 3^-, 4^+, 5^+, 6^+, 7^-) = T_1^D + T_2^D + T_{4a}^D + T_{4b}^D + T_{4c}^D\,, \tag{B.7}
$$

with

$$
\begin{aligned}
T_{1}^{D} =& -\frac{[6|P_{71}|3\rangle^2\,\langle 3\,2\rangle\,[1\,6]^2}{[6\,7]\,[7\,1]\,\langle 3\,4\rangle\,\langle 4\,5\rangle\,[1|P_{671}|5\rangle[6|P_{671}|2\rangle t_{671}} \times \left(\frac{[6|P_{71}|3\rangle}{[1\,6]\,\langle 3\,2\rangle}\right)^{2h}, \\
T_{2}^{D} =& -\frac{\langle 7\,1\rangle\,\langle 7\,2\rangle^2\,[6|P_{712}|3\rangle^3}{\langle 1\,2\rangle\,\langle 3\,4\rangle\,\langle 4\,5\rangle\,[6|P_{712}|2\rangle\langle 5|P_{34}P_{12}|7\rangle t_{345}t_{712}} \times \left(\frac{\langle 7\,1\rangle}{\langle 7\,2\rangle}\right)^{2h}, \\
T_{4a}^{D} =& \frac{[4|P_{56}|7\rangle^3\,\langle 7\,2\rangle^2\,\langle 7\,1\rangle^2}{\langle 5\,6\rangle\,\langle 6\,7\rangle\,\langle 7\,1\rangle\,\langle 1\,2\rangle\,[3\,4]\,[3|P_{12}|7\rangle\langle 7|P_{56}P_{34}|2\rangle\langle 7|P_{12}P_{34}|5\rangle} \times \left(\frac{\langle 7\,1\rangle}{\langle 7\,2\rangle}\right)^{2h}, \\
T_{4b}^{D} =& \frac{\langle 2\,3\rangle\,[1|P_{56}|7\rangle^2\langle 7|P_{56}P_{234}|3\rangle^2}{\langle 5\,6\rangle\,\langle 6\,7\rangle\,\langle 3\,4\rangle\,[1|P_{567}|5\rangle[1|P_{234}|4\rangle\langle 7|P_{56}P_{234}|2\rangle t_{234}t_{567}} \times \left(\frac{-\langle 7|P_{56}P_{234}|3\rangle}{\langle 2\,3\rangle\,[1|P_{56}|7\rangle}\right)^{2h}, \\
T_{4c}^{D} =& \frac{[1|P_{123}|7\rangle^2[2|P_{123}|7\rangle^2}{\langle 4\,5\rangle\,\langle 5\,6\rangle\,\langle 6\,7\rangle\,[1\,2]\,[2\,3]\,[3|P_{123}|7\rangle[1|P_{123}|4\rangle t_{123}} \times \left(\frac{-[2|P_{123}|7\rangle}{[1|P_{123}|7\rangle}\right)^{2h}.
\end{aligned}
\tag{B.8}
$$

$$
E : A_7(s_1, \bar{s}_2, 3^+, 4^-, 5^-, 6^+, 7^+) = T_{1a}^E + T_{1b}^E + T_{1c}^E + T_2^E + T_3^E + T_4^E\,, \tag{B.9}
$$

with

$$
\begin{aligned}
T^E_{1a} &= \frac{\langle 2\,4\rangle^2\, t_{671}\langle 1|P_{67}P_{23}|4\rangle^2}{\langle 6\,7\rangle\,\langle 7\,1\rangle\,\langle 2\,3\rangle\,\langle 3\,4\rangle\,[5|P_{671}|1\rangle[5|P_{671}|2\rangle\langle 6|P_{71}P_{23}|4\rangle t_{234}}\Big(\frac{-\langle 1|P_{67}P_{23}|4\rangle}{\langle 2\,4\rangle\, t_{671}}\Big)^{2h},\\
T^E_{1b} &= \frac{-[3|P_{71}|6\rangle^2\,[2\,3]\,\langle 4\,5\rangle^3\,\langle 1\,6\rangle^2}{\langle 5\,6\rangle\,\langle 6\,7\rangle\,\langle 7\,1\rangle\,[2|P_{71}|6\rangle[3|P_{45}|6\rangle\langle 6|P_{71}P_{23}|4\rangle\langle 1|P_{23}P_{45}|6\rangle}\Big(\frac{-[2\,3]\,\langle 6\,1\rangle}{[3|P_{71}|6\rangle}\Big)^{2h},\\
T^E_{1c} &= \frac{[3|P_{345}|2\rangle^2[3|P_{345}|1\rangle^2}{\langle 6\,7\rangle\,\langle 7\,1\rangle\,\langle 1\,2\rangle\,[3\,4]\,[4\,5]\,[5|P_{345}|2\rangle[3|P_{345}|6\rangle t_{345}}\Big(\frac{[3|P_{345}|1\rangle}{[3|P_{345}|2\rangle}\Big)^{2h},\\
T^E_2 &= \Big(\frac{\langle 4\,5\rangle}{\langle 3\,5\rangle}\Big)^4 T^B_2\,,\\
T^E_3 &= \frac{[7|P_{456}|1\rangle^2[7|P_{456}|2\rangle^2\,\langle 4\,5\rangle^3}{\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 5\,6\rangle\,[7|P_{456}|3\rangle[7|P_{56}|4\rangle\langle 1|P_{23}P_{45}|6\rangle t_{123}t_{456}}\times\Big(\frac{[7|P_{456}|1\rangle}{[7|P_{456}|2\rangle}\Big)^{2h},\\
T^E_4 &= \Big(\frac{\langle 1\,4\rangle^2\,\langle 2\,4\rangle^2}{\langle 1\,3\rangle^2\,\langle 2\,3\rangle^2}\Big)\Big(\frac{\langle 2\,3\rangle\,\langle 1\,4\rangle}{\langle 1\,3\rangle\,\langle 2\,4\rangle}\Big)^{2h} T^B_4\,.
\end{aligned}
\tag{B.10}
$$

$$
F: A_7(s_1,\bar{s}_2,3^+,4^-,5^+,6^-,7^+) = T^F_{1a}+T^F_{1b}+T^F_{1c}+T^F_2+T^F_3+T^F_4\,, \tag{B.11}
$$

with

$$
\begin{aligned}
T^F_{1a} &= \frac{[5|P_{234}|4\rangle^2\,\langle 2\,4\rangle^2\,[5|P_{71}|6\rangle^2\,\langle 1\,6\rangle^2}{\langle 6\,7\rangle\,\langle 7\,1\rangle\,\langle 2\,3\rangle\,\langle 3\,4\rangle\,[5|P_{234}|1\rangle[5|P_{234}|2\rangle\langle 6|P_{71}P_{23}|4\rangle t_{671}t_{234}}\Big(\frac{[5|P_{234}|4\rangle\,\langle 1\,6\rangle}{\langle 2\,4\rangle\,[5|P_{234}|6\rangle}\Big)^{2h},\\
T^F_{1b} &= \frac{-\,[2\,3]^2\,[3|P_{71}|6\rangle^2\,\langle 4\,6\rangle^4\,\langle 1\,6\rangle^2}{\langle 6\,7\rangle\,\langle 7\,1\rangle\,[2|P_{71}|6\rangle\,[2\,3]\,\langle 4\,5\rangle\,\langle 5\,6\rangle\,[3|P_{45}|6\rangle\langle 6|P_{71}P_{23}|4\rangle\langle 1|P_{23}P_{45}|6\rangle}\Big(\frac{-\,[2\,3]\,\langle 6\,1\rangle}{[3|P_{71}|6\rangle}\Big)^{2h},\\
T^F_{1c} &= \frac{\langle 6\,2\rangle^2\,\langle 1\,6\rangle^2\,[3\,5]^4}{\langle 6\,7\rangle\,\langle 7\,1\rangle\,\langle 1\,2\rangle\,[3\,4]\,[4\,5]\,[5|P_{345}|2\rangle[3|P_{345}|6\rangle t_{345}}\Big(\frac{\langle 6\,1\rangle}{\langle 6\,2\rangle}\Big)^{2h},\\
T^F_2 &= \frac{\langle 4\,6\rangle^4}{\langle 3\,6\rangle^4} T^C_2\,,\\
T^F_3 &= \frac{\langle 4\,6\rangle^4\,[7|P_{123}|1\rangle^2[7|P_{123}|2\rangle^2}{\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 4\,5\rangle\,\langle 5\,6\rangle\,[7|P_{123}|3\rangle[7|P_{123}|4\rangle\langle 1|P_{23}P_{45}|6\rangle t_{123}t_{456}}\times\Big(\frac{[7|P_{123}|1\rangle}{[7|P_{123}|2\rangle}\Big)^{2h},\\
T^F_4 &= \frac{\langle 1\,4\rangle^{2+2h}\,\langle 2\,4\rangle^{2-2h}}{\langle 1\,3\rangle^{2+2h}\,\langle 2\,3\rangle^{2-2h}} T^C_4\,.
\end{aligned}
\tag{B.12}
$$

Appendix C

Solutions of Quadruple Cuts

In the generalised Unitarity method, the quadruple cuts each define a unique box integral coefficient [78]. In the framework of 4-dimensional Unitarity, these cuts take a particularly simple form since the four momentum constraints serve to eliminate all four orders of loop integration and thus "freeze" the integral, leaving the problem purely one of algebraically solving the resulting momentum constraints to obtain the (semi) unique solution for the loop momenta.

The quadruple cuts divide into five main categories, depending upon how many of the corners consist of a single external particle, known as a "massless" corner due to the fact that for massless particles, the momentum for the corner obeys the constraint $k^2 = 0$, which has a major effect on the resulting solution. The other case is a corner with more than one external particle, known as a "massive" corner as in this case $K^2 \neq 0$.

The five cases are thus referred to as the four-mass box, the three-mass box, two possible two-mass configurations known as the "hard" and "easy" configurations, and the one-mass box.

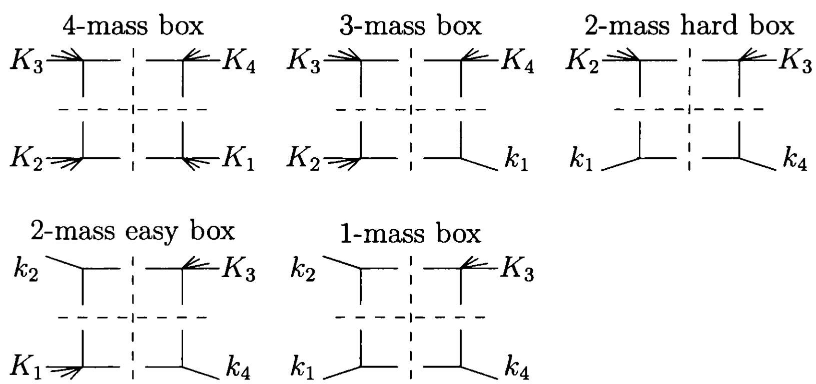


Figure C.1: The 5 basic classes of quadruple cut

In principle, one could conceive of a valid "no-mass" box; however, this cut would only appear in the very specific case of the four-point loop. As with the double and triple cuts, we can only consider helicity configurations which result in valid amplitudes at all four corners. Thus, for example in a scalar loop, the case with an amplitude $A^{tree}(k_a^+, k_b^+, l_i, l_j)$ at any corner is trivially zero since this is a vanishing tree amplitude. In addition, we can exploit the properties of Generalized Unitarity to gain additional constraints: Since a valid quad cut must be obtainable by cutting an additional leg on valid triple cut, which must in turn be obtainable by cutting an additional leg on a valid double cut, we can additionally conclude that diagrams in which the four individual corners are non-zero, but the "sub-total" amplitude obtained by considering two or three adjacent corners together is vanishing, are also zero.

We can use these conditions both to determine which momentum channels yield valid quad cuts, and to determine the possible helicities of the loop particles. Note that in some cases there is some ambiguity; for example, for the boxes appearing in a gluon loop, there are typically two valid choices for the internal helicities. These represent the two possible solutions for the loop momentum, and thus the full cut is given by the sum of the two cases, since the loop integral must reduce to leave only these two points in momentum space satisfying the constraints.

The two types of quad cuts we are interested in however are those with a $\mathcal{N}=1$ chiral multiplet loop, and the complex scalar loop. In this first case this means that the only possible quad cuts are those in which we obtain a valid solution for both helicity choices of a gluino circulating, and for a complex scalar, such that we have a full multiplet circulating. We can then exploit the same cancellation within the multiplet that appears in the double and triple cuts to reduce the overall power of the cut in l. Meanwhile, the scalar loop quad cuts involve only those cuts which yield a valid solution for a scalar circulating in the loop.

We can now consider how to obtain the loop momentum solutions for the various cases required.

C.1 1-mass box

For the scalar loop, we have the configuration

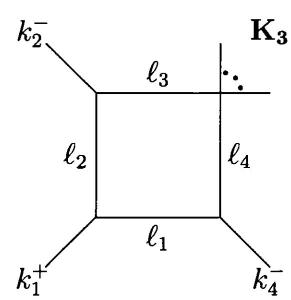


Figure C.2: General 1-mass box, $(-+-)$ configuration

The key is to apply the constraints on the four loop momenta given by the cutting, specifically $l_i^2 = 0$ for all four sides, and apply momentum conservation. We can begin by solving for l_1, using the property $\not{l}_2 = \not{l}_1 + \not{k}_1$. The constraint $l_2^2 = 0$ then yields $(l_1 + k_1)^2 = l_1^2 + 2l_1.k_1 + k_1^2 = 0$, which in turn gives $\langle l_1\, k_1\rangle\, [k_1\, l_1] = 0$, since k_1 is a massless corner. This implies that either $\lambda_{l_1} \propto \lambda_{k_1}$, or $\tilde{\lambda}_{l_1} \propto \tilde{\lambda}_{k_1}$.

We note that the solution $[k_1\, l_1] = 0$ would result in the k_1 corner amplitude being zero, since it is an $\overline{\text{MHV}}$ amplitude and thus contains the numerator factor $[k_1\, l_1]^2$. The only nontrivial solution thus occurs if we make the choice $\langle l_1\, k_1\rangle = 0$, and thus obtain $\lambda l_1 = \alpha_1 \lambda_{k_1}$. Applying the same principle at the other three massless corners, we can solve for six of the eight momentum spinors,

$$\begin{aligned} \lambda_{l_1} &= \lambda_{k_1}, & \tilde{\lambda}_{l_1} &= z\tilde{\lambda}_{k_4}, \\ \lambda_{l_2} &= \lambda_{k_1}, & \tilde{\lambda}_{l_2} &= w\tilde{\lambda}_{k_2}, \\ \tilde{\lambda}_{l_3} &= \beta_3\tilde{\lambda}_{k_2}, & \lambda_{l_4} &= \beta_4\lambda_{k_1}. \end{aligned} \tag{C.1}$$

The remaining two loop momentum spinors can be solved using momentum conservation. We can solve λ_{l_3} using

$$\begin{aligned} \not{l}_3 =& \not{l}_2 + \not{k}_2\,, \\ \lambda_{l_3}\tilde{\lambda}_{l_3} =& w\lambda_{k_1}\tilde{\lambda}_{k_2} + \lambda_{k_2}\tilde{\lambda}_{k_2}\,, \\ =& (w\lambda_{k_1} + \lambda_{k_2})\tilde{\lambda}_{k_2}\,. \end{aligned} \tag{C.2}$$

We thus obtain $\lambda_{l_3} = (w\lambda_{k_1} + \lambda_{k_2})$. A similar analysis of l_1 and l_4 yields the spinor $\lambda_{l_4} = (z\lambda_{k_1} - \lambda_{k_4})$. All that remains is to determine the constant coefficients z and w, which again can be done using momentum conservation,

$$\begin{aligned} \not{l}_3 =& \not{l}_1 + \not{k}_1 + \not{k}_2\,, \\ l_3^2 =& 0 = \langle 1\,2\rangle\,[2\,1] + z[4|P_{12}|1\rangle\,, \\ =& [2\,1] - z\,[4\,2]\ , \\ z =& -\frac{[1\,2]}{[4\,2]}\,. \end{aligned} \tag{C.3}$$

Similarly for w,

$$\begin{aligned} \not{l}_4 =& \not{l}_2 - \not{k}_1 - \not{k}_4\,, \\ 0 =& \langle 1\,4\rangle\,[4\,1] + w\,[2\,4]\,\langle 1\,4\rangle\ , \\ w =& \frac{[4\,1]}{[4\,2]}\,. \end{aligned} \tag{C.4}$$

Note that with these solutions the spinors λ_{l_3} and λ_{l_4} can be written in a form in which their on-shell nature is more explicit,

$$\begin{aligned}\lambda_{l_3} =& \frac{[1\,4]}{[2\,4]}\lambda_{k_1} + \lambda_{k_2}\,,\\ =& \frac{[1\,4]\,\lambda_{k_1} + [2\,4]\,\lambda_{k_2}}{[2\,4]}\,,\\ =& \frac{P_{12}|4]}{[2\,4]}\,.\end{aligned} \tag{C.5}$$

We therefore have a full solution for the loop momenta for this box,

$$\begin{array}{ll}\lambda_{l_1} = \lambda_{k_1}\,, & \tilde{\lambda}_{l_1} = -\frac{[1\,2]}{[4\,2]}\tilde{\lambda}_{k_4}\,,\\ \lambda_{l_2} = \lambda_{k_1}\,, & \tilde{\lambda}_{l_2} = \frac{[4\,1]}{[4\,2]}\tilde{\lambda}_{k_2}\,,\\ \lambda_{l_3} = \frac{P_{12}|4]}{[2\,4]}\,, & \tilde{\lambda}_{l_3} = \tilde{\lambda}_{k_2}\,,\\ \lambda_{l_4} = \frac{P_{14}|2]}{[2\,4]}\,, & \tilde{\lambda}_{l_4} = \tilde{\lambda}_{k_4}\,,\end{array} \tag{C.6}$$

Note that the equivalent flipped-helicity case,

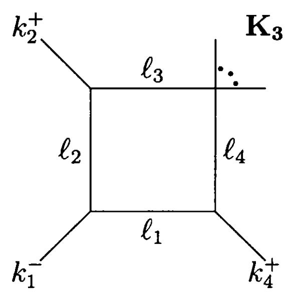


Figure C.3: General 1-mass box, $(+ - +)$ configuration

can be solved by simply conjugating the above momentum solutions, due to the symmetry between the diagrams; since this result is independent of the structure of the K_3 corner, these two momentum solutions apply to all 1-mass boxes we will encounter.

C.2 2-mass "hard" box

We consider first the configuration in figure C.4, where we label the upper propagator as l_P to denote its special status due to being adjacent to two massive corners. As in the 1-mass case we can exploit the massless corners to solve loop momentum spinors in terms of the external momenta. This yields four of the eight spinors,

$$\begin{array}{ll}\lambda_{l_1} = \lambda_{k_1}\,, & \tilde{\lambda}_{l_1} = z\tilde{\lambda}_{k_4}\,,\\ \lambda_{l_2} = \alpha_2\lambda_{k_1}\,, & \tilde{\lambda}_{l_4} = \beta_4\tilde{\lambda}_{k_4}\,.\end{array} \tag{C.7}$$

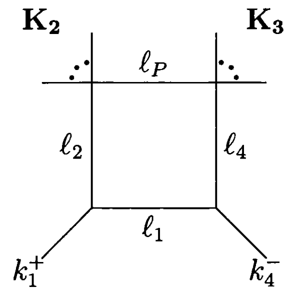


Figure C.4: General 2-mass hard box

One can next solve for z using the property $\not{l}_P = \not{l}_1 + \not{P}$, where $P = k_1 + K_2 = -k_4 - K_3$,

$$\begin{aligned} l_P^2 = 0 =& P^2 + z[4|P|1\rangle\,, \\ z =& -\frac{P^2}{[4|P|1\rangle}\,. \end{aligned} \tag{C.8}$$

This leaves only the spinors λ_{l_P} and $\tilde{\lambda}_{l_P}$ to be solved, since as with the 1-mass the spinors λ_{l_4} and $\tilde{\lambda}_{l_2}$ can be solved by momentum conservation. To try to separate the positive and negative chirality on-shell spinors for l_P we contract l_P in the form $l_P = l_1 + P$ with some reference spinors λ_α and $\tilde{\lambda}_\beta$,

$$\begin{aligned} [\beta|l_P|\alpha\rangle =& [\beta|P|\alpha\rangle - \frac{P^2\,[\beta\,4]\,\langle 1\,\alpha\rangle}{[4|P|1\rangle}\,, \\ =& \frac{[\beta|P|\alpha\rangle[4|P|1\rangle - [\beta|PP|4]\,\langle 1\,\alpha\rangle}{[4|P|1\rangle}\,, \\ =& \frac{[\beta|P|1\rangle[4|P|\alpha\rangle}{[4|P|1\rangle}\,, \end{aligned} \tag{C.9}$$

where the last line is obtained by applying the Schouten identity. We can thus identify the on-shell spinors as $\lambda_{l_P} = \frac{[4|P}{[4|P|1\rangle}$ and $\tilde{\lambda}_{l_P} = P|1\rangle$. This thus gives us the complete solution to the box,

$$\begin{aligned} \lambda_{l_1} &= \lambda_{k_1}\,, & \tilde{\lambda}_{l_1} &= -\tfrac{P^2}{[4|P|1\rangle}\tilde{\lambda}_{k_4}\,, \\ \lambda_{l_2} &= \lambda_{k_1}\,, & \tilde{\lambda}_{l_2} &= -\tfrac{[4|PK_2}{[4|P|1\rangle}\,, \\ \lambda_{l_P} &= \tfrac{[4|P}{[4|P|1\rangle}\,, & \tilde{\lambda}_{l_P} &= P|1\rangle\,, \\ \lambda_{l_4} &= \tfrac{K_3P|1\rangle}{[4|P|1\rangle}\,, & \tilde{\lambda}_{l_4} &= \tilde{\lambda}_{k_4}\,. \end{aligned} \tag{C.10}$$

As with the 1-mass box, the flipped helicity case for the massless legs can be solved by simply conjugating the above solution.

C.3 2-mass "easy" box

In this case we have the general structure

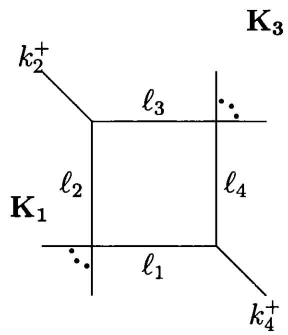


Figure C.5: The general 2-mass easy box

We can use the massless corners to fix four of the spinors,

$$\begin{aligned} \lambda_{l_1} &= \alpha_1 \lambda_{k_4}, \quad \lambda_{l_2} = \alpha_2 \lambda_{k_2}, \\ \lambda_{l_4} &= \alpha_4 \lambda_{k_4}, \quad \lambda_{l_3} = \alpha_3 \lambda_{k_2}. \end{aligned} \tag{C.11}$$

Next we note that we can express the proportionality coefficients in the above in terms of the unknown spinors, e.g.

$$\begin{aligned} \not{l}_2 =& \not{l}_1 + \not{K}_1 \,, \\ 0 =& K_1^2 + \alpha_1 [l_1|K_1|4\rangle \,, \\ \alpha_1 =& - \frac{K_1^2}{[l_1|K_1|4\rangle} \,. \end{aligned} \tag{C.12}$$

As with the hard case, one can try to infer from this the spinors for l_2 by contracting with some reference spinors λ_A and $\tilde{\lambda}_B$,

$$\begin{aligned} [B|l_1|A\rangle =& [B|K_1|A\rangle - \frac{K_1^2 \langle A\,4\rangle \, [Q\,B]}{[Q|K_1|4\rangle} \,, \\ =& - \frac{[B|K_1|4\rangle [Q|K_1|A\rangle}{[Q|K_1|4\rangle} \,. \end{aligned} \tag{C.13}$$

From this one can solve for l_2,

$$\lambda_{l_2} = \lambda_2, \quad \tilde{\lambda}_{l_2} = \frac{\langle 4|K_1}{\langle 2\,4\rangle} \,. \tag{C.14}$$

One can thus obtain the full solution for the box coefficient,

$$
\begin{aligned}
\lambda_{l_1} &= \lambda_{k_4}, & \tilde{\lambda}_{l_1} &= -\tfrac{\langle 2|K_1}{\langle 2\,4\rangle}, \\
\lambda_{l_2} &= \lambda_{k_2}, & \tilde{\lambda}_{l_2} &= \tfrac{\langle 4|K_1}{\langle 2\,4\rangle}, \\
\lambda_{l_3} &= \lambda_{k_2}, & \tilde{\lambda}_{l_3} &= -\tfrac{\langle 4|K_3}{\langle 2\,4\rangle}, \\
\lambda_{l_4} &= \lambda_{k_4}, & \tilde{\lambda}_{l_4} &= \tfrac{\langle 2|K_3}{\langle 2\,4\rangle}.
\end{aligned}
\tag{C.15}
$$

C.4 3-mass box

In principle one can obtain 4-mass boxes from cuts, however the smallest number of external particles which gives rise to such cuts is 8; hence, the most difficult class of box to appear in a 7-point calculation is the 3-mass box (at 6-point, even these do not appear, and in this case the 2-mass boxes and 1-mass box solve the complete set of box coefficients).

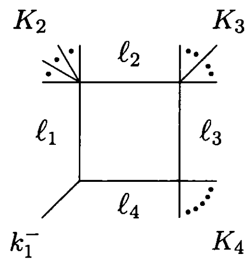


Figure C.6: 3-mass box with MHV corner

The same tricks we used to solve the 2-mass easy box can be reapplied here. We use the massless corner to fix $\tilde{\lambda}_{l_1}$ and $\tilde{\lambda}_{l_4}$, and use momentum conservation on the K_2 and K_4 corners to fix the constant coefficient,

$$
\begin{aligned}
\tilde{\lambda}_{l_1} &= -\frac{K_2^2}{[1|K_2|l_1\rangle}\tilde{\lambda}_{k_1}, \\
\tilde{\lambda}_{l_4} &= \frac{K_4^2}{[1|K_4|l_4\rangle}\tilde{\lambda}_{k_1}.
\end{aligned}
\tag{C.16}
$$

Next we can apply momentum conservation on these corners again to try to isolate the λ spinors,

$$
\begin{aligned}
\not{l}_2 =& \not{l}_4 - (\not{K}_3 + \not{K}_4)\,, \\
0 =& (K_3 + K_4)^2 - \frac{K_4^2[1|(K_3 + K_4)|l_4\rangle}{[1|K_4|l_4\rangle}\,, \\
=& (K_3 + K_4)^2[1|K_4|l_4\rangle - K_4^2[1|(K_3 + K_4)|l_4\rangle\,, \\
=& [1|K_4(K_3 + K_4)(K_3 + K_4)|l_4\rangle - [1|K_4K_4(K_3 + K_4)|l_4\rangle\,, \\
=& [1|K_4K_3(K_3 + K_4)|l_4\rangle\,,
\end{aligned} \tag{C.17}
$$

allowing us to obtain

$$
\begin{aligned}
\lambda_{l_4} =& \frac{(K_3 + K_4)K_3K_4|1]}{[1|K_3K_4|1]}\,, \\
\lambda_{l_1} =& \frac{[1|K_2K_3(K_2 + K_3)}{[1|K_3K_2|1]}\,.
\end{aligned} \tag{C.18}
$$

To solve the sides with two adjacent massive corners (l_2 and l_3), we use the same technique as used to solve the 2-mass hard box; we contract the momentum with arbitrary spinors α and β, and use the Schouten identity to isolate the on-shell spinors,

$$
\begin{aligned}
[\beta|l_3|\alpha\rangle =& [\beta|(K_2 + K_3)|\alpha\rangle - \frac{[\beta\,1]\,[1|K_2K_3(K_2 + K_3)|\alpha\rangle}{[1|K_3K_2|1]}\,, \\
=& \frac{[\beta|K_2 + K_3|\alpha\rangle[1|K_3K_2|1] - [\beta\,1]\,[1|K_2K_3(K_2 + K_3)|\alpha\rangle}{[1|K_3K_2|1]}\,, \\
=& \frac{[\beta|K_3K_2|1][1|K_2 + K_3|\alpha\rangle}{[1|K_3K_2|1]}\,.
\end{aligned} \tag{C.19}
$$

We thus identify the spinors,

$$
\begin{aligned}
\lambda_{l_3} =& [1|(K_2 + K_3)\,, \\
\tilde{\lambda}_{l_3} =& \frac{K_3K_2|1]}{[1|K_3K_2|1]}\,.
\end{aligned} \tag{C.20}
$$

The derivation for l_2 is identical, and we obtain the full solution,

$$
\begin{array}{rlrl}
\lambda_{l_1} = & \frac{[1|K_2K_3(K_2+K_3)}{[1|K_2K_3|1]}, & \tilde{\lambda}_{l_1} = & \tilde{\lambda}_{k_1}, \\
\lambda_{l_2} = & [1|(K_3 + K_4), & \tilde{\lambda}_{l_2} = & \frac{K_3K_4|1]}{[1|K_3K_4|1]}, \\
\lambda_{l_3} = & [1|(K_2 + K_3), & \tilde{\lambda}_{l_3} = & \frac{K_3K_2|1]}{[1|K_2K_3|1]}, \\
\lambda_{l_4} = & \frac{(K_3+K_4)K_3K_4|1]}{[1|K_3K_4|1]}, & \tilde{\lambda}_{l_4} = & \tilde{\lambda}_{k_1}.
\end{array} \tag{C.21}
$$

As before, the solution for the case with k_1 of positive helicity can be trivially obtained by conjugating this solution.

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